Prelim Workshop
UC Berkeley

Linear Algebra 2

August 10, 2022

Outline

Canonical Forms

- Similarity.
- Finitely-generated modules over a PID.
- Rational and Jordan canonical forms, uniqueness.

More on Diagonalizability

- $A \in M_n(F)$ diagonalizable if and only if minimal polynomial splits in F with no repeated roots.
- $A \in M_n(F)$ is triangularizable over F if and only if minimal polynomial splits in F.
- Commuting matrices.

More on Inner Products

- Spectral theorems (real and complex case).
- Gram-Schmidt.

Problems

7.7.10. Let A and B be two real $n \times n$ matrices. Suppose there is a complex invertible $n \times n$ matrix U such that $A = UBU^{-1}$. Show that there is a real invertible $n \times n$ matrix V such that $A = VBV^{-1}$. (In other words, if two real matrices are similar over $\mathbb{C}$, then they are similar over $\mathbb{R}$).

7.6.24 Find the eigenvalues, eigenvectors, and the Jordan canonical form of

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

considered as a matrix in $\mathbb{Z}/3\mathbb{Z}$.

7.6.30. Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

7.7.6. Let A and B be $n \times n$ matrices over a field F such that $A^2 = A$ and $B^2 = B$. Suppose that A and B have the same rank. Prove that A and B are similar.

7.5.17. Let S be a nonempty commuting set of $n \times n$ complex matrices ($n \geq 1$). Prove that the members of S have a common eigenvector. (Sidenote: As an exercise use this fact to prove the important fact that if A and B are diagonalizable matrices such that $AB = BA$, then there is an invertible matrix T such that TAT^{-1} and TBT^{-1} are both diagonal)

7.6.17. Let V be a finite-dimensional vector space and $T : V \to V$ a diagonalizable linear transformation. Let $W \subset V$ be a linear subspace which is mapped to itself by T. Show that the restriction of T to W is diagonalizable.