We are studying linear maps

$$T: V \longrightarrow W$$

where V + W are vector spaces over a field F.

Def - A set $v_1, \dots, v_n \in V$ is linearly independent if whenever $c_1, \dots, c_n \in F$
s.t.

$$c_1 v_1 + \dots + c_n v_n = 0 \implies c_1 = \dots = c_n = 0$$

- A basis for V is a linearly independent set which also spans
- Dimension of V is the length of any basis

Assume unless otherwise stated vector spaces are finite dimensional.

Two facts:

i) Any linearly independent set can be extended to a basis.

ii) Any spanning set has a subset which is a basis.

Most important fact: If $T: V \longrightarrow W$ linear, then

$$\dim V = \underbrace{\dim(\mathrm{Ker}(T))}_{\text{also known as nullity}} + \underbrace{\dim(\mathrm{Im}(T))}_{\text{Also known as rank}}$$

This implies when $\dim V = \dim V$, injectivity $\Leftrightarrow$ surjectivity $\Leftrightarrow$ invertible

<u>7.1.1</u> Idea is that evaluating a polynomial at a point is a linear map

$$\{\text{polynomials}\} \longrightarrow F$$

So

1. This is the kernel of

$$\underbrace{\{\text{polys degree} \le 3\}}_{\text{dim } 4} \longrightarrow F$$
$$p \longmapsto p(1)$$

Thus dim of the kernel is $4-1=3$, thus 4 polynomials in the kernel are automatically linearly independent

2. Take $1, 1+x, 1+x^2, 1+x^3$, counterexample.

Remember, the λ-eigenspace of a linear map $T: V \to V$ is

$$\text{Ker}(T - \lambda I)$$

T is <u>diagonalizable</u> if there exists a basis $v_1, \ldots, v_n$ of V s.t. v_i is an eigenvector for T for each i.

For any linear map $T: F^n \to F^m$ there is a $m \times n$ matrix A s.t.

$$T(x) = Ax \quad \text{for all } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in F^n$$

A is given by

$$\left(T(e_1) \middle| \cdots \middle| T(e_n) \right) \qquad e_i := \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th place}$$

$T: F^n \to F^n$ is diagonalizable $\iff$ there is a diagonal matrix D and an invertible matrix P s.t.

$$D = P^{-1} A P$$

(useful for computing powers of A)

In this case

$$P = \left(v_1 \middle| \cdots \middle| v_n \right) \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

v_i a basis of eigenvectors $\qquad$ λ_i the v_i eigenvalue.

An <u>inner product</u> on a $\boxed{\text{real}}$ vector space V is an operation

$$\langle , \rangle : V \times V \to \mathbb{R} \quad \text{s.t.}$$

i) (Bilinear) $\langle -, v \rangle$ and $\langle w, - \rangle$ are linear maps $V \to \mathbb{R}$ for all $v, w \in V$

ii) (symmetric) $\langle v, w \rangle = \langle w, v \rangle$

iii) (positive definite) $\langle v, v \rangle \geq 0$, equality $\iff$ $v = 0$.

If $T: V \to W$ linear, its <u>adjoint</u> is the map

real

$$T^*: W \to V$$

determined by

$$\langle Tv, w \rangle = \langle v, T^* w \rangle \quad \text{for all } v \in V,\ w \in W$$

If $T: \mathbb{R}^n \to \mathbb{R}^m$ has matrix A, then the matrix of $T^*: \mathbb{R}^m \to \mathbb{R}^n$ is A^T.

<u>Thm</u> (Spectral Thm) $T: \mathbb{R}^n \to \mathbb{R}^n$ has an orthonormal basis (i.e. $\langle v_i, v_j \rangle = \delta_{ij}$) of eigenvectors of T $\iff$ $T = T^*$

<u>symmetric or self-adjoint</u>

<u>7.2.7</u> T is symmetric and thus diagonalizable. Note $2 \Rightarrow 1$ so we show each eigenspace has dimension at most 1.

We are solving the system of equations $Tx = \lambda x$. First equation is

$$a_1 x_1 + b_1 x_2 = \lambda x_1$$

$b_1 \neq 0$ so can solve x_2 in terms of x_1. Suppose we can solve $x_2, \ldots, x_i$ in terms of x_1. Then i^{th} equation is

$$b_{i-1} x_{i-1} + a_i x_i + b_i x_{i+1} = \lambda x_i$$

since $b_i \neq 0$ can solve for x_{i+1} in terms of x_1.

Last equation is

$$b_{n-1} x_{n-1} + a_n x_n = \lambda x_n$$

If we can solve for x_n, then λ an eigenvalue and eigenspace only depends on one parameter x_1, thus eigenspace 1-dim.
Otherwise not an eigenvalue

The determinant is computed as follows:

$$\det(a) = a$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Then for A an $n \times n$ matric, define A_{ij} to be the submatrix w/ ith row and jth row deleted.
Then fix some row i, then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \underset{\substack{\uparrow \\ (i,j)\text{ entry of } A}}{a_{ij}} \det(A_{ij})$$

It doesn't matter which row we choose

E.g. $\det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} - \det\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

Important properties of the determinant of an $n \times n$ matrix A

i) Multi-linear, alternating in the columns

$\Rightarrow$ $\det A$ is unchanged by adding a column to another

ii) $\det(AB) = \det(A)\det(B)$

iii) $\det A \neq 0 \iff A$ invertible

iv) $\det A = \det A^T$

<u>7.2.11</u> A is called an $n \times n$ <u>vandermonde matrix</u>

By induction. Holds for

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$$

Then in $n \times n$ case, let A_i denote A with the ith column and last row deleted. Then

$$(*) \quad \det A = \pm \left((\det A_n) x_n^{n-1} + \dots + (\det A_2) x_n + \det A_1 \right)$$

$(n-1)$ degree poly in x_n, has at most $(n-1)$ roots. Setting $x_n = x_i$ $\det A = 0$ since A then has two identical rows. Thus

$$\det A = c \prod_{i=1}^{n-1} (x_n - x_i) \qquad c \in \mathbb{R}$$

From $(*)$, $c = \det A_n$, A_n is an an $(n-1) \times (n-1)$ vander monde so

$$\det A = \prod_{j < i \leq n-1} (x_i - x_j) \prod_{i=1}^{n-1} (x_n - x_i)$$

as desired. (need a sign argument)

7.2.12

1. If T is not invertible, then there is $x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \neq 0$ s.t.

$$Tx = 0$$

thus

$$x_n a_i^n + \cdots + x_1 a_i + x_0 = 0$$

for each i. Thus the n degree poly w/ coeffs x_n has roots $a_0, \ldots, a_n$, only possible if they are not distinct.

2. T is invertible, so for any $b = \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix}$ there is a unique $x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$ s.t.

$$Tx = b$$

that is,

$$x_n a_i^n + \cdots + x_1 a_i + x_0 = b_i$$

for each i.

Note that if A is an $n \times n$ triangular matrix, then $\det A = $ (product of the diagonal elements)

Thus, a triangular matrix is invertible $\iff$ all diagonal entries nonzero.

Now, let A be an $n \times n$ matrix (values can be in F a field). Then A satisfies a polynomial

$p(x) = a_n x^n + \dots + a_1 A + a_0 I \in F[x]$ if

$$p(A) := a_n A^n + \dots + a_1 A + a_0 I = 0$$

The set $\{ p \in F[x] \mid p(A) = 0 \}$ is an ideal in $F[x]$. $F[x]$ is a pid, thus there is a unique monic polynomial $m_A \in F[x]$ s.t.

$$\{p \in F[x] \mid p(A) = 0\} = (m_A)$$

m_A is called the <u>minimal polynomial</u> of A

The <u>characteristic polynomial</u> $p_A \in F[x]$ is

$$\det(A - xI)$$

so λ is an eigenvalue of $A \iff p_A(\lambda) = 0$.

Here are some important facts about these two

i) (Cayley-Hamilton) $m_A \mid p_A$, so in particular

$$p_A(A) = 0$$

ii) p_A divides some power of m_A, so p_A and m_A have the same irreducible factors up to powers (so λ an eigenvalue $\iff m_A(\lambda) = 0$)

iii) If A is $n \times n$, p_A is always degree n. $\text{Trace}(A) = \{n\text{-}1 \text{ coefficient of } p_A\}$

$\det(A) = \{\text{constant term of } p_A\}$

7.5.3 $A^m = 0$ thus $m_A \mid x^m$

But this gives that $m_A = x^k$ for some $k \le n$

(one way to see $k \le n$ is $m_A \mid p_A$). Thus $A^n = 0$.

7.6.5 The characteristic polynomial of A is

$$x^3 - 8x^2 + 20x - 16 = (x-4)(x-2)^2$$

thus the minimal polynomial is either $(x-4)(x-2)^2$ or $(x-4)(x-2)$. We can see that it is $(x-4)(x-2)$ directly, by either checking $(A-4I)(A-2I) = 0$ or checking that $\dim(\text{Ker}(A-2I)) = 2$.

Now, by the Euclidean algorithm

$$(\star) \quad x^{10} = p(x)\, m_A(x) + ax + b \qquad a, b \in \mathbb{R}$$

Thus $\quad A^{10} = p(A)\, m_A(A) + aA + bI = aA + bI$

Plugging $x = 2$ and $x = 4$ into $(\star)$ we have

$$2^{10} = 2a + b$$

$$4^{10} = 4a + b$$

solving this linear system gives $\quad a = 2^9(2^{10} - 1)$
$\quad b = -2^{11}(2^9 - 1)$

$$A^{10} = aA + bI = \begin{pmatrix} 3a+b & a & a \\ 2a & 4a+b & 2a \\ -a & -a & a+b \end{pmatrix}$$

One last example computing determinant:

Find determinant of an $n \times n$ matrix of the form:

$$\begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \ddots & \vdots \\ b & b & a & \ddots & \vdots \\ b & b & \ddots & \ddots & b \\ \vdots & \vdots & \ddots & b & a \\ b & b & \cdots & b & \end{pmatrix}$$

Adding a row/column to another row/column does not change det so add all columns to first column to get

$$\begin{pmatrix} a+(n-1)b & b & b & \cdots & b \\ a+(n-1)b & a & b & \ddots & \vdots \\ \vdots & b & a & \ddots & \vdots \\ \vdots & b & \ddots & \ddots & b \\ \vdots & \vdots & \ddots & b & a \\ a+(n-1)b & b & \cdots & b & \end{pmatrix}$$

then subtract the first row from the others to get

$$\begin{pmatrix} a+(n-1)b & b & \cdots & b & b & b \\ & a-b & 0 & \cdots & \cdots & 0 \\ & & a-b & 0 & \cdots & 0 \\ & & & \ddots & & \vdots \\ & & & & \ddots & \vdots \\ & & & & & a-b \end{pmatrix}$$

so upper triangular, so determinant is

$$(a-b)^{n-1}\left(a+(n-1)b\right)$$