- 10 sessions, practice prelim
- prelim archive, prelim lecture, Berkeley problems in math

in prelim resource Folder.

$(\mathbb{Z}/2^a\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{a-2}\mathbb{Z}$

$(\mathbb{Z}/p^a\mathbb{Z})^\times \cong \mathbb{Z}/p^{a-1}(p-1)\mathbb{Z}$ p odd

- $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/p_1^{a_1}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_k^{a_k}\mathbb{Z})^\times$
- $G = H \rtimes K \iff H \trianglelefteq G,\ H \cap K = \{1\},\ HK = G$
- $|\mathrm{Orb}(x)| = \dfrac{|G|}{|\mathrm{Stab}(x)|}$

$|\{g \cdot x : g \in G\}|$ $\quad \{g \in G : g \cdot x = x\}$

$\sum_x' |\mathrm{Orb}(x)| = |X|$

special case $G \circlearrowright G$ by conjugations

$$|G| = \sum_x' |\text{conj classes}(x)| = \sum_x' \frac{|G|}{|C_G(x)|} = |Z(G)| + \sum_x' \frac{|G|}{|C_G(x)|}$$

If $|G| = n$, then G cyclic $\Leftarrow$ At most one subgroup of every order $d \mid n$.

- At most one subgroup of order d, then at most $\varphi(d)$ elements of order d.
  $\hookrightarrow$ number of generators of a cyclic group of order d.
  $$n = |G| = \sum_d' \text{elements of order } d \le \sum_d' \varphi(d) = n$$
- $|G| = p_1^{a_1} \cdots p_k^{a_k}$, exactly one Sylow p_i-subgroup, necessarily normal
  $$G = P_1 \times \cdots \times P_k$$
  $$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}$$ so, reduce to p-groups.

Look at element $g \in G$ of maximal order, Take $h \notin \langle g \rangle$, h has order dividing order of g (G is a p-grp) $\langle g \rangle$ will have a subgroup of order $|\langle h \rangle|$

$|G| = p^k$

$|H| = p^{k-1}$

p^{k-2}

$\vdots$

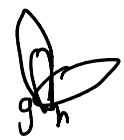
Take $x \notin H$.

$|H| = p^{k-1}$ — p^k

p^3

p^2

$\langle g \rangle$ ... of order $|\langle h \rangle|$



1

G finite group

$X = \{(g,h) \in G \times G : gh = hg\}$

$|X| = c|G|, \quad c = \#$ conj classes

Solution: $|X| = \sum_{g \in G} |\{h \in G : gh = hg\}| = \sum_{g \in G} |C_G(g)|$

$= \sum_{g \in G} \frac{|G|}{|\text{conj class of } g|}$

Burnside's lemma

$= |G| \underbrace{\sum_{g \in G} \frac{1}{|\text{conj class of } g|}}_{\text{each conj class contributes } 1} = |G| \cdot c$

$= |G| \sum_{e} \sum_{g \in e} \frac{1}{|e|} = |G| \cdot c$

Determine $|X|$ for $G = S_5$.

conj classes in $S_5 \longleftrightarrow$ partitions of 5

$7 \begin{cases} 1+1+1+1+1 \\ 1+1+1+2 \\ 1+2+2 \\ 1+1+3 \\ 2+3 \\ 1+4 \\ 5 \end{cases}$

$|X| = 7 \cdot 5! = 840$

If $|G| = 30$, then G has a cyclic subgroup H, $|H| = 15$.

Apply Sylow's thms, $p = 5$. $\quad n_5 | 30, \quad n_5 \equiv 1 \pmod 5$

$n_5 = 1$ or 6.

$p = 3 \quad n_3 | 30, \quad n_3 \equiv 1 \pmod 3$

$n_3 = 1$ or 10

You can't have $\underbrace{n_5 = 6}_{\to 24 \text{ elements of order } 5}$ and $\underbrace{n_3 = 10}_{\to 20 \text{ elements of order}}$

If $P_3 \trianglelefteq G$, then take preimage of a Sylow 5-subgroup of $\underbrace{G/P_3}_{\text{order } 10}$ under $G \twoheadrightarrow G/P_3$.

(Same if $P_5 \trianglelefteq G$).

If $|H| = 15$, H has P_3, P_5, $P_5 \trianglelefteq H$ by Sylow

$H = P_5 \rtimes P_3$

$\mathbb{Z}/3\mathbb{Z} \cong P_3 \longrightarrow \operatorname{Aut}(P_5) \cong \operatorname{Aut}(\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$

trivial so $H = P_3 \times P_5 \cong \mathbb{Z}/15\mathbb{Z}$.

Alternate approach:

$\ldots (P_3)\}$

Alternate approach:
Find $P_3 \le N_G(P_5)$
take $H = P_3 P_5$

$\mathbb{Z}/3\mathbb{Z} \cong P_3 \xrightarrow[\text{trivial}]{} Aut(P_5) \cong Aut(\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$
so $H = P_3 \times P_5 \cong \mathbb{Z}/15\mathbb{Z}$.

$H \le G$ of index n, then G has a normal subgroup $N \trianglelefteq G$ of index $\le n!$ with $N \le H$.

$G \curvearrowright \{gH\}$ $\quad g \cdot (g'H) = gg'H$

Get $G \xrightarrow{\varphi} S_n$ with kernel N satisfying
$[G:N] = |\text{im } \varphi| \le n!$

If $g \in N$, then $g \cdot (g'H) = g'H$ for all $g'H$
$gH = H \Rightarrow g \in H$.

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1) If G non-abelian, then $G/Z(G)$ is not cyclic.

pf: Assume $G/Z(G) = \langle g Z(G) \rangle$

Then each $g'Z(G) = g^k Z(G)$ for some $k \in \mathbb{Z}$

Then each $g' \in G$ can be written as $g^k z$ for $k \in \mathbb{Z}$, $z \in Z(G)$

$$(g^k z)(g^{k'} z') = (g^{k'} z')(g^k z).$$

2) If $|G| = p^n$, then $Z(G)$ nontrivial (if $n > 0$).

$$\underbrace{|G|}_{\text{div by } p} = |Z(G)| + \sum_x \underbrace{\frac{|G|}{|C_G(x)|}}_{\text{larger than } 1,}$$
$\Rightarrow$ div by p because these are the sizes of the non-central conjugacy classes

Then $p \mid |Z(G)|$
so $|Z(G)| \ne 1$.

3) If $|G| = p^2$, then G abelian
- $|Z(G)| = 1$ (impossible by (2))
- $|Z(G)| = p$ (impossible by (1), because then $|G/Z(G)| = p^2/p = p$ so cyclic).
- $|Z(G)| = p^2 \Rightarrow$ done

- A_6 does not have an index 3 subgroup

pf: We get $A_6 \xrightarrow{\varphi} S_3$ with kernel $\le$ this index 3 subgroup

A_6 simple, so φ is either trivial or injective
impossible since

pf. We ...

A_6 simple, so φ is either trivial or injective

impossible   impossible since $|A_6| > |S_3|$

- No simple groups of order 120.

Sylow's theorems $\Rightarrow$ $n_5 \mid 24$, $n \equiv 1 \pmod 5$

$\Rightarrow n_5 = 1$ or 6, so $n_5 = 6$ by simplicity

We get $G \to S_6$ (either act on 6 Sylow 5-subgroups or use $[G : N_G(P_5)] = 6$).

Look at $G \to S_6 \xrightarrow{\text{sgn}} \{\pm 1\}$, must be trivial by simplicity of G.

so image of G is contained in A_6.

The map $G \to A_6$ is injective by simplicity of G and nontriviality of $G \to S_6$.

But A_6 has no subgroups of index 3.