Explicit Solutions for Some DEs

A *linear ordinary differential equation* is of the form

$$a_n(x) f^{(n)}(x) + \dots + a_1(x) f'(x) + a_0(x) f(x) = g(x)$$

where the $a_i : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are given.

Sometimes we stipulate that f and derivatives take certain values at certain points, called *initial conditions*.

It gets the name linear since

$$f \longmapsto a_n(x) f^{(n)} + \dots + a_1(x) f' + a_0(x) f$$

is a linear map.

If $g \equiv 0$ the linear ODE is called *homogeneous*, otherwise *inhomogeneous*.

Homogeneous equations, since kernel is a subspace, have the <u>superposition property</u> i.e. if f_1 + f_2 are solutions to the homogeneous equation, then so is

$$c_1 f_1 + c_2 f_2 \quad \text{for all } c_1, c_2 \in \mathbb{R}$$

Now, if $g \neq 0$ then a solution of

$$a_n(x) f^{(n)}(x) + \dots + a_1(x) f'(x) + a_0(x) f(x) = g(x)$$

is called a <u>particular solution</u>. If f_1 + f_2 are particular solutions, then $f_1 - f_2$ is a solution to the homogeneous equation. Thus the <u>general solution</u> is

$$f_p(x) + f_h(x)$$

where f_p is a particular solution and f_h

is any homogenous solution. Thus to solve:

i) Find the homogeneous solutions

ii) Find one particular solution

iii) Combine.

An illustrative example are second order constant coefficient equations

$$f'' + af' + bf = g(x)$$

Assuming we've found a particular solution f_p (educated guess)

We just need to solve homogeneous equation

$$f'' + af' + bf = 0$$

Find roots of

$$r^2 + ar + br$$

If distinct roots λ_1, λ_2 the all solutions are

$$C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$ (basis for homog solutions)

If repeated root λ

$$c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

Spring 2008 4A: First solve homogeneous equation. Roots of $r^2 - 2r - 1$ are

$$1 \pm \sqrt{2}$$

Thus all homogeneous solutions are

$$c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}$$

To find particular solution, guess Ae^{-x} giving

$$Ae^{-x} + 2Ae^{-x} - Ae^{-x} = e^{-x}$$

$$\Rightarrow \quad A + 2A - A = 1$$

$$\Rightarrow \quad 2A = 1 \quad \Rightarrow \quad A = \frac{1}{2}$$

So general solution is

$$\frac{1}{2}e^{-x} + C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x}$$

The initial condition gives

$$C_1 + C_2 = -1/2$$
$$(1+\sqrt{2})C_1 + (1-\sqrt{2})C_2 = 1/2$$

which has a unique solution. ✓

A linear system of first order ODEs is an equation of the form.

$$\begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = A(t)\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

where is an $n \times n$ matrix of given functions, and the b_i are given as well.

If the $b_i \equiv 0$ it is <u>homogeneous</u>. The principle of superposition applies, and a general solution is

$$x_p(t) + x_h(t)$$

For a constant matrix A which is diagonalizable, the homogeneous solutions are

$$C_1 e^{\lambda_1 t} v_1 + \dots + C_n e^{\lambda_n t} v_n$$

where v_i is a λ_i-eigenvector.

<u>Fall 2021 6A</u> This matrix has really nice properties. It is symmetric so it is orthogonally diagonalizable. The char poly is

$$\det(A - \lambda I) = \lambda^2 (14 - \lambda)$$

thus 0-eigenspace is dim 2, 14 eigenspace is dim 1.

0-eigenspace: $A = (u_1 | u_2 | u_3)$ note $u_3 = 3u_1$, $u_2 = 2u_1$

so $\text{Span}\left(\begin{pmatrix}3\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\0\end{pmatrix}\right)$

14-eigenspace: Since its symmetric looking for

$$3x_1 - x_3 = 0 \Rightarrow x_1 = \frac{1}{3}x_3$$

$$2x_1 - x_2 = 0 \Rightarrow x_1 = \frac{1}{2}x_2$$

so $\text{Span}\begin{pmatrix}1\\2\\3\end{pmatrix}$

General solution is

$$y(t) = C_1 v_1 + C_2 v_2 + C_3 e^{14t} v_3$$

Then $y(0) = C_1 v_1 + C_2 v_2 + C_3 v_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$

So solve the linear system.

A couple more techniques. Consider 1 diff eq of the form

$$f'(x) = g(x)\, h(f(x))$$

more suggestively, $y = f(x)$, then

$$\frac{dy}{dx} = g(x)\, h(y)$$

$$\Rightarrow \quad \frac{1}{h(y)} dy = g(x) dx$$

"separation of variables"

then integrate both sides

Ex.

$$\frac{dy}{dx} = xy$$

$$\Rightarrow \quad \frac{1}{y} dy = x\, dx$$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C$$

$$\Rightarrow \quad y = ke^{x^2/2}$$

Another technique is called integrating factors.

This is for equations of the form

$$f' + p(x) f = q(x)$$

In this case take $r(x) = \int p$, then multiply by the integrating factor

$$\left(e^{r(x)} f(x)\right)' = e^{r(x)}\left(f'(x) + p(x) f(x)\right)$$
$$= e^{r(x)} q(x)$$

Then integrating we get

$$f(x) = Ce^{-r(x)} + e^{-r(x)}\int \left(e^{r(x)} q(x)\right)$$

Fall 2021 1A

a) $y_1(t) = t$ works

b) If $y(t)$ is any solution, then

$$u(t) = y(t) - y_1(t) \quad \text{is s.t.}$$

$$\begin{aligned} u' &= y' - 1 \\ &= y^2 - ty \\ &= (y-t)^2 + ty - t^2 \\ &= (y-t)^2 + t(y-t) \\ &= u^2 + tu \end{aligned}$$

So u solves

$$u' = u^2 + tu$$

$$\iff \quad \frac{u'}{u^2} - \frac{t}{u} = 1$$

Set $z = \frac{-1}{u}$ then $z' = \frac{u'}{u^2}$

Then $$z' + tz = 1$$

Ok, now apply integrating factor $e^{\int t} = e^{t^2/2}$

$$\Rightarrow \quad z(t) = Ce^{-t^2/2} + e^{-t^2/2}\int e^{t^2/2}$$

$$\Rightarrow \quad u(t) = \frac{1}{Ce^{-t^2/2} - e^{-t^2/2}\int e^{t^2/2}}$$

Now we cover some facts which give local existence & uniqueness in certain cases.

<u>Def</u> A function $f: U \subseteq \mathbb{R} \to \mathbb{R}$ is called <u>Lipschitz continuous</u> if there exists a constant $L \geq 0$ s.t. for all $x, y \in U$

$$|f(x) - f(y)| \leq L|x-y|$$

the smallest possible L is called the <u>Lipschitz constant.</u>

<u>Thm</u> (Picard-Lindelöf) Suppose $f(t,y)$ is uniformly Lipschitz in y and continuous in t. Then for some $\varepsilon > 0$, the IVP

$$\frac{dy}{dt} = f(t, y(t)), \quad y(0) = y_0$$

has a unique solution solution on some interval $(-\varepsilon, \varepsilon)$

even stronger $\longrightarrow$ If $M_{a,b}$ the maximum of f on $[-a,a] \times [y_0-b, y_0+b]$ and L the lipschitz constant in y, a unique solution exist in $[-\delta, \delta]$ for any $\delta < \min\left(\frac{b}{M_{a,b}}, \frac{1}{L}\right)$.

3.1.3 Fall 1993 14 Existence of IVP

$$\frac{dy}{dx} = y^n$$

Separate variables: $\frac{1}{y^n} dy = dx$

$$\Rightarrow \frac{1}{(1-n) y^{n-1}} = x + C$$

$$\Rightarrow \frac{1}{y^{n-1}} = (1-n)x + C \quad \Rightarrow C = 1/y(0)^{n-1} > 0$$

Then

$$y = \frac{1}{(C-(n-1)x)^{1/(n-1)}}$$

This solve the equation with $y(0) = C^{-1/(n-1)} > 0$ in $[0, \frac{C}{n-1})$. This blows up as $x \nearrow \frac{C}{n-1}$ but is the unique solution by Picard, so none such exist.

<u>Thm</u> (Implicit function) If $f(x,y)$ is a continuously differentiable function, $f(a,b)=0$, $\frac{\partial f}{\partial y}(a,b)=0$. Then there exists a neighborhood U of a and a unique continuously differentiable function g s.t. $g(a)=b$ and

$$f(x,g(x))=0 \quad \text{in } U$$

<u>Thm</u> (Inverse function) If $f(x)$ is cont diff ble and $f'(a)\neq 0$, then f has a cont diffable inverse $f^{-1}(y)$ in a neighborhood of a w/

$$\left(f^{-1}(y)\right)' = \frac{1}{f'(f^{-1}(y))}$$

3.1.10 - Fall 1982

1. f not necessarily Lipschitz in y so can't apply P-L

Instead look at

$$\frac{dx}{dy} = \frac{1}{f(y)}$$

cont in y, Lipschitz in x (no x dependence) then apply PL w/ intial condition $x(c)=0$. to get a local solution

Then apply inverse function theorem.

For 2., we need the solution to

$$\frac{dx}{dy} = \frac{1}{f(y)}, \quad x(c) = 0$$

to go to $\pm\infty$ to the right, $\mp\infty$ to the left.

Need

$$\int_c^{\infty} \frac{1}{f(y)}\,dy \qquad \int_{-\infty}^{c} \frac{1}{f(y)}\,dy$$

to diverge.

Gronwall's inequality is about extracting information about a function from a differential inequality.

<u>Summer 1982 6</u> (Also appeared on 2021 exam)

If we consider $(e^{-x}f)' = e^{-x}f' - e^{-x}f$

integrating factor from $f' - f > 0$

$$= e^{-x}(f' - f) > 0$$

$\Rightarrow e^{-x}f$ is strictly increasing.

$e^{-x}f$ is zero at x_0, then $e^{-x}f > 0$ for all $x > x_0 \Rightarrow f > 0$ for all $x > x_0$.

We can generalize this arguement.

<u>Thm</u> (Gronwall's Inequality) If η is a differentiable function on $[0, T]$, ϕ integrable s.t.

$$\eta'(t) \le \phi(t)\,\eta(t)$$

Then

$$\eta(t) \le e^{\int_0^t \phi(s)\,ds}\,\eta(0)$$

pf Take derivative of $\left(\eta(t)\,e^{-\int_0^t \phi(s)\,ds}\right)$

then apply the assumptions $\square$

In particular if $\eta' \leq \phi \eta$, $\eta(0) = 0$, then

$$\eta \equiv 0.$$