Complex Analysis 2

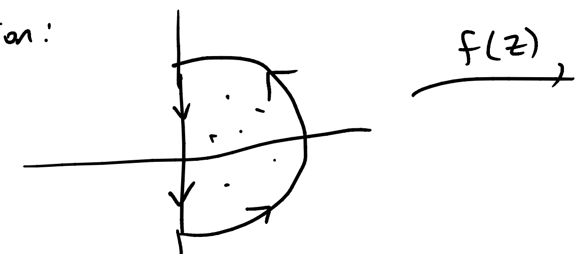
Argument principle: Net change in argument $= 2\pi \cdot (\underbrace{\#\text{zeros} - \#\text{poles}}_{\text{with multiplicity}})$

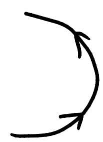
$f(z)$

$\#$ roots of $z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$ ($n \geq 1$, $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \neq 0$)

in $\{Re(z) > 0\} = \begin{cases} n & n \text{ even} \\ n-1 & n \text{ odd} \end{cases}$

Solution:



On semicircle , argument of $f(z)$ increases by $2n\pi$

$z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$

$z = iy$

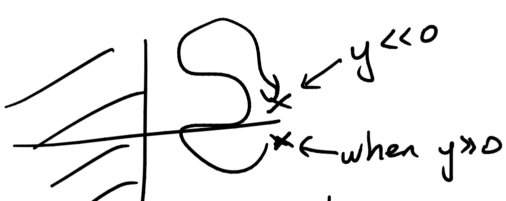
$(iy)^{2n} + \alpha^2 (iy)^{2n-1} + \beta^2$

n even

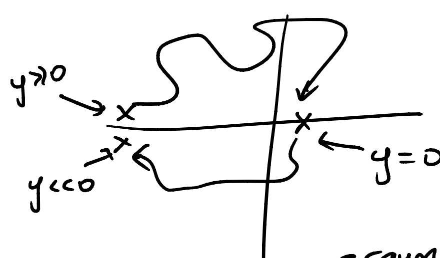
$y^{2n} - i\alpha^2 y^{2n-1} + \beta^2$

n odd

$-y^{2n} + i\alpha^2 y^{2n-1} + \beta^2$



no change in argument



argument decreases by 2π

total change: $2\pi n$

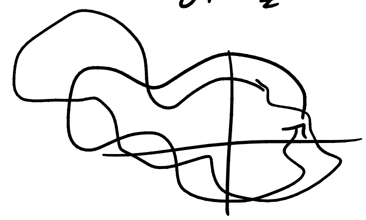
$\boxed{n}$

total change: $2\pi n - 2\pi$

$\boxed{n-1}$

Rouché's Theorem: If $|f| < |g|$ on $\partial\Omega$,

Rouché's Theorem: If $|f|<|g|$ on $\partial\Omega$,
then g and $f+g$ have the same number of zeros in Ω (counting multiplicity).



Ex: $z^5+z^3+5z^2+2$ how many roots in $1<|z|<2$

If $|z|=1$, $|5z^2|=5>4 \geq |z^5+z^3+2|$) $\Rightarrow$ 2 zeros in $|z|\leq 1$

triangle inequality

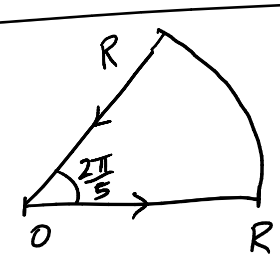
If $|z|=2$ $|z^5|=32>\underset{8+20+2}{\overset{}{30}} \geq |z^3+5z^2+2|$) $\Rightarrow$ 5 zeros in $|z|<2$

Number of zeros $=5-2=3$

Residue Theorem: $\displaystyle\int_{\partial\Omega} f(z)\,dz = 2\pi i \sum_{z_i} \mathrm{Res}_{z=z_i} f(z)$

$\mathrm{Res}_{z=z_i} f(z) = a_{-1}$ coefficient of the Laurent expansion $\sum_n a_n(z-z_i)^n$.

$$\int_0^\infty \frac{1}{1+x^5}\,dx$$



$$f(z) = \frac{1}{1+z^5}$$

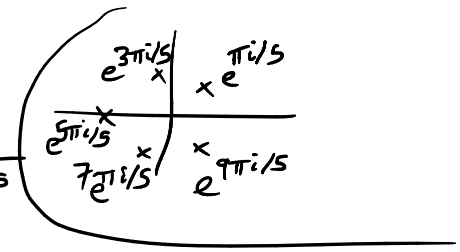
$$\int_\Gamma f(z)\,dz = \underbrace{\int_0^R f(z)\,dz}_{\substack{z=t \\ \int_0^R f(t)\,dt}} + \underbrace{\int_R^{Re^{2\pi i/5}} f(z)\,dz}_{\substack{\to 0 \\ \text{as } R\to\infty \\ \text{Length} \sim R \\ \text{Max} \sim R^{-5} \\ \leq \frac{\frac{2\pi}{5}R}{R^5-1}}} + \underbrace{\int_{Re^{2\pi i/5}}^{0} f(z)\,dz}_{-\underbrace{\int_0^{Re^{2\pi i/5}} f(z)\,dz}_{z=e^{2\pi i/5}t}}$$

$$\int^R f(e^{2\pi i/5}t)\,e^{2\pi i/5}\,dt$$

$$\leq \frac{2\frac{\pi}{5}R}{R^5-1} \to 0 \qquad \int_0^R f(e^{2\pi i/5}t)e^{2\pi i/5}\,dt$$

$$\int_0^\infty \frac{1}{1+x^5}\,dx - \int_0^\infty e^{2\pi i/5}\frac{1}{1+x^5}\,dx = (1-e^{2\pi i/5})\int \frac{1}{1+x^5}\,dx$$

$f(z) = \dfrac{1}{1+z^5}$ has simple poles



$$(1-e^{2\pi i/5})\int_0^\infty \frac{1}{1+x^5}\,dx = 2\pi i \operatorname{Res}_{z=e^{\pi i/5}} \frac{1}{1+z^5}$$

$$1+z^5 = a_1(z-e^{\pi i/5}) + a_2(z-e^{\pi i/5})^2 + \cdots$$

$$\frac{1}{1+z^5} = a_1^{-1}(z-e^{\pi i/5})^{-1} + \text{constant term} + \cdots$$

$m=1$

$(a_m z^m + a_{m+1}z^{m+1} + \cdots$

$(a_n z^n + a_{n+1}z^{n+1}\cdots$

$\underline{a_{m+n}z^{m+n}\cdots}$

$m+n=0$

$$a_1 = \frac{d}{dz}(1+z^5)\Big|_{e^{\pi i/5}} = 5e^{4\pi i/5}$$

$$(1-e^{2\pi i/5})\int_0^\infty \frac{1}{1+x^5}\,dx = 2\pi i\,\frac{1}{5e^{4\pi i/5}}$$

$$\int_0^\infty \frac{1}{1+x^5}\,dx = \frac{2\pi i}{5e^{4\pi i/5}(1-e^{2\pi i/5})}$$

$$= \frac{2\pi i}{-5e^{-\pi i/5}(1-e^{2\pi i/5})}$$

$$= \frac{2\pi i}{5(e^{\pi i/5}-e^{-\pi i/5})} \qquad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

$$= \frac{\pi}{5\sin\left(\frac{\pi}{5}\right)}$$

$0<a<b$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|ae^{i\theta}-b|^4}\,d\theta \qquad \int_{|z|=1} f(z)\,dz$$

$z=e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$$\frac{1}{|ae^{i\theta}-b|^4} = \frac{1}{(ae^{i\theta}-b)^2(ae^{-i\theta}-b)^2}$$

$e^{2i\theta}$

$$= \frac{1}{\ } \int \frac{z}{\ }\,dz$$

$$\frac{\quad}{|ae^{i\theta}-b|^4} = \frac{\quad}{(ae^{i\theta}-b)^2(ae^{-i\theta}-b)^2}$$

$$= \frac{e^{2i\theta}}{(ae^{i\theta}-b)^2(a-be^{i\theta})^2}$$

$$= \frac{1}{2\pi i}\int_{|z|=1} \frac{z}{(az-b)^2(a-bz)^2}\,dz$$

$$= \operatorname{Res}_{z=\frac{a}{b}} \frac{z}{(az-b)^2(a-bz)^2}$$

$$f(z) = a_{-2}\left(z-\frac{a}{b}\right)^{-2} + a_{-1}\left(z-\frac{a}{b}\right)^{-1} + \dots$$

$$\left.\left(\frac{d}{dz}\left(z-\frac{a}{b}\right)^2 f(z)\right)\right|_{z=\frac{a}{b}}$$

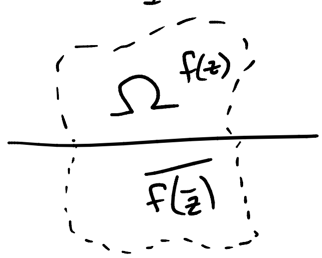
A lot of algebra.

$$\frac{a^2+b^2}{(a^2-b^2)^3}$$

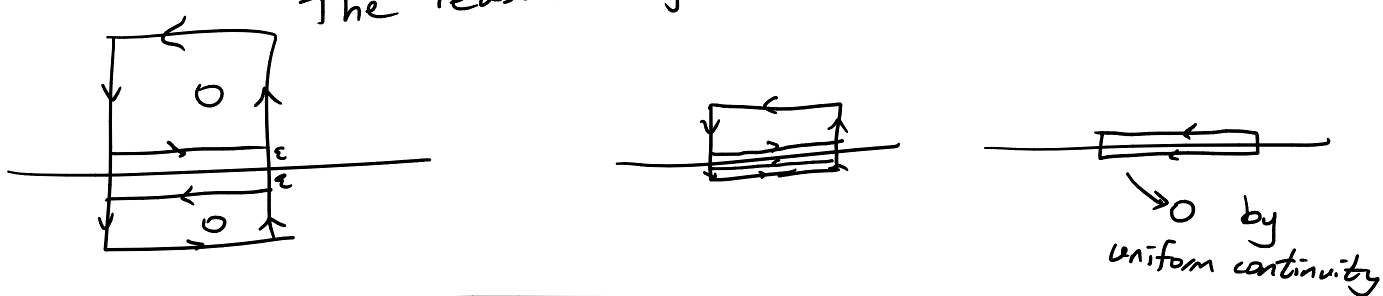
Schwarz reflection:

Key idea: If $f(z)$ analytic, then so is $\overline{f(\bar{z})}$

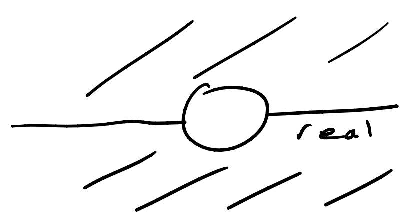
If $f(z)$ analytic in $\Omega \subseteq \mathbb{H}$, and extends continuously to $\partial\Omega \cap \mathbb{R}$ with real values then $f(z)$ extends analytically to the reflection of Ω.



The reason why $f(z)$ is analytic on $\partial\Omega \cap \mathbb{R}$



If $f(z)$ analytic on $|z|>1$, and real on $(1,\infty)$, then $f(z)$ real on $(-\infty,-1)$



$\overline{f(\bar{z})}$ agrees with $f(z)$ on $(1,\infty)$.

Identity theorem (if $f(z)$, $g(z)$ agree on a set with an accumulation point inside the domain, then $f=g$)

$\Rightarrow f(z) = \overline{f(\bar{z})}$

Identity theorem … with an accumulation point inside the domain, then $f=g$)

$\Rightarrow f(z)=\overline{f(\bar{z})}$

$\Rightarrow f(z)$ is real on the real axis.

If $f(z)$ analytic on $|z-a|<r$ extends continuously on $|z-a|\le r$, then does $f(z)$ extend analytically to $|z-a|<r+\delta$?

No! $f(z)=\sqrt{z}$