Week 3: Subgroups, normality, and quotient groups

Practice Problems

1. This problem looks scary but in fact it is easy (and useful).

Let G be a group and let H be a subgroup of G. We define the normalizer and centralizer of H as

$$N_G(H) = \{ g \in G : gHg^{-1} = H \},$$

 $C_G(H) = \{ g \in G : gh = hg \text{ for all } h \in H \}.$

- (a) Show that $N_G(H)$ is a subgroup of G.
- (b) Show that $C_G(H)$ is a normal subgroup of $N_G(H)$.
- (c) Show that H is a normal subgroup of G if and only if $N_G(H) = G$.
- (d) Let K be a subgroup of G containing H. Show that H is normal in K if and only if K is a subgroup of $N_G(H)$. In other words, $N_G(H)$ is the largest subgroup of G in which H is normal.
- 2. Let G be the dihedral group of order 8, let $r \in G$ be a reflection, and let $s \in G$ be rotation by 180°. Let $H = \{1, r\}$ and let $N = \{1, r, s, rs\}$. Show that
 - H is a normal subgroup of N,
 - N is a normal subgroup of G,
 - H is not a normal subgroup of G.

This shows that normality is not transitive.

3. Let G and H be groups. Show that $(G \times H)/H \cong G$.

Presentation Problems

- 1. Let G be a group and let H be a subgroup of G.
 - (a) Construct a homomorphism $N_G(H) \to \operatorname{Aut}(H)$ with kernel $C_G(H)$.
 - (b) Construct an injective homomorphism $N_G(H)/C_G(H) \to \operatorname{Aut}(H)$.

We define the center of G as

$$Z(G) = C_G(G) = \{g \in G : gh = hg \text{ for all } h \in G\}.$$

By practice problem 1(b), $Z(G) = C_G(G)$ is a normal subgroup of $G = N_G(G)$.

- (c) Show that $\operatorname{Inn}(G) = \{h \mapsto ghg^{-1} : g \in G\}$ is a normal subgroup of $\operatorname{Aut}(G)$.
- (d) Show that $G/Z(G) \cong \text{Inn}(G)$.
- (e) Show that if G/Z(G) is cyclic then G is abelian.
- 2. The quaternion group is the set

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

with the relations $i^2 = j^2 = k^2 = -1$, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j.

- (a) Determine the subgroup lattice of Q_8 .
- (b) Show that every subgroup of Q_8 is normal.
- (c) Determine the center of Q_8 .

- (d) Let x and y be two random (uniformly distributed and independently chosen) elements of Q_8 . Determine the probability that x and y commute.
- 3. Let G be a group and let A and B be subgroups of G. Show that the elementwise product

$$AB = \{ab : a \in A, b \in B\}$$

is a subgroup of G if and only if AB = BA.

- 4. Let G be a group. For $g, h \in G$, the *commutator* of g and h is the element $[g, h] = ghg^{-1}h^{-1}$. Let [G, G] be the subgroup of G generated by commutators [g, h] for $g, h \in G$.
 - (a) Show that [G, G] is a normal subgroup of G and that the quotient $G^{ab} = G/[G, G]$ is abelian.
 - (b) Show that if $\varphi \colon G \to A$ is a homomorphism from G to an abelian group A then $[G, G] \subseteq \ker \varphi$ and there exists a unique homomorphism $\widetilde{\varphi} \colon G^{ab} \to A$ so that the diagram



commutes.

- (c) Show that [G, G] is the smallest normal subgroup of G with abelian quotient.
- (d) Construct an interesting homomorphism $\operatorname{Out}(G) \to \operatorname{Aut}(G^{\operatorname{ab}})$ where $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$ is the outer automorphism group of G.

Warning: Out(G) is not a subset of Aut(G).

Tricky Problems

- 1. Let G be a nonabelian finite group. Let x and y be two random (uniformly distributed and independently chosen) elements of G. Let p be the probability that x and y commute.
 - (a) Show that $p \leq 5/8$.
 - (b) Show that p = 5/8 if and only if $G/Z(G) \cong C_2 \times C_2$.

You may use these two facts which we will prove next week:

- If H is a subgroup of G then |H| divides |G|.
- If H is a normal subgroup of G then |G|/|H| = |G/H|.
- 2. Let G be a finite group of order n and let S be a nonempty subset of G. Show that the n-fold product

$$S^n = \underbrace{S \cdots S}_n = \{s_1 \cdots s_n : s_1, \dots, s_n \in S\}$$

is a subgroup of G. Hint: Consider the size of S^k as k increases.