Week 9: Galois groups of polynomials (14.6)

Practice Problems

1. Compute the Galois group of $x^3 - 2x + 4$ over $\mathbb{Q}$.
2. Compute the Galois group of $x^3 - x + 1$ over $\mathbb{Q}$.
3. Compute the Galois group of $x^3 + x^2 - 2x - 1$ over $\mathbb{Q}$.

Presentation Problems

1. Let p be a prime and let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree p with exactly $p-2$ real roots. Show that the Galois group of $f(x)$ is isomorphic to S_p.
2. Compute the Galois group of $x^5 + 4x^2 - 5x - 3$ over $\mathbb{Q}$.
3. Compute the Galois group of $x^4 - 2$ over $\mathbb{Q}$.
4. Compute the Galois group of $x^4 + 8x + 12$ over $\mathbb{Q}$.

Group Theory Problem

1. Let G be a finite group and let A be a normal Hall subgroup of G, meaning that A is a normal subgroup of G such that $\gcd(|A|, [G : A]) = 1$.

   (a) Show that if A is abelian then $G \cong A \rtimes (G/A)$. *Hint*: Consider the cohomology group $H^2(G/A, A)$.

   (b) Show that if A is solvable then $G \cong A \rtimes (G/A)$. *Hint*: Consider the commutator subgroup of A.

   (c) Show that if $N_G(P) = G$ for every Sylow subgroup P of A then $G \cong A \rtimes (G/A)$. *Hint*: Nilpotent.

   (d) Show that $G \cong A \rtimes (G/A)$. *Hint*: If $N_G(P) \neq G$ for some Sylow subgroup P of A then inductively assume that the result holds for the normal Hall subgroup $N_G(P) \cap A$ of $N_G(P)$. You may find the Frattini argument $G = N_G(P)A$ helpful.

   This is known as the Schur-Zassenhaus theorem.

Tricky Problems

1. Consider the polynomials $p_1(x) = x^5 - 2x^4 + x^3 + x^2 - x + 1$ and $p_2(x) = x^5 - x - 1$. Let K_j be the splitting field of $p_j(x)$ over $\mathbb{Q}$. Compute $\mathrm{Gal}(K_j/\mathbb{Q})$. Show that there exists a unique quadratic extension $F_j/\mathbb{Q}$ with $F_j \subseteq K_j$. What is this quadratic extension?
2. Let a and n be positive integers. Suppose that a is squarefree and that $a \nmid n$. Show that the Galois group of $x^n - a$ is isomorphic to $\mathbb{Z}/n\mathbb{Z} \rtimes_\varphi (\mathbb{Z}/n\mathbb{Z})^\times$.