Week 4: Polynomial rings (9.1, 9.2)

Let R be a commutative ring with identity. Let K be a field.

Practice Problems

1. Factor the polynomial $x^4 + 1$ in the rings $\mathbb{C}[x]$, $\mathbb{R}[x]$, $\mathbb{Q}[x]$.
2. Construct a surjective ring homomorphism $K[x, y] \to K$ with kernel (x, y). Construct a surjective ring homomorphism $K[x, y] \to K[y]$ with kernel (x). Deduce that (x, y) is maximal and that (x) is prime.
3. Show that $\mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/(x^2 + 5)$.

Presentation Problems

1. Show that $K[x]$ contains infinitely many primes. *Hint*: Look at Euclid's proof that there are infinitely many primes in $\mathbb{Z}$.
2. Let $I = (xy, (x - y)z) \subseteq K[x, y, z]$. Show that $\sqrt{I} = (xy, xz, yz)$.
3. (a) Show that $K[x, y]/(y^2 - x) \cong K[y]$.

   (b) Show that $K[x, y]/(y^2 - x) \not\cong K[x, y]/(y^2 - x^2)$.
4. (a) Construct an injective ring homomorphism $K[x, y]/(xy) \to K[x] \times K[y]$.

   (b) Show that $K[x, y]/(xy) \not\cong K[x] \times K[y]$.

Module Theory Problem

For this problem, do not assume that R is commutative.

1. Let $\cdots \to P_1 \to P_0 \to M \to 0$ and $0 \to N \to Q_0 \to Q_1 \to \cdots$ be exact sequences of R-modules. Suppose that each P_i is projective, meaning that $\operatorname{Hom}_R(P_i, -)$ takes exact sequences of R-modules to exact sequences of abelian groups. Suppose that each Q_i is injective, meaning that $\operatorname{Hom}_R(-, Q_i)$ takes exact sequences of R-modules to exact sequences of abelian groups. The sequence $\cdots \to P_1 \to P_0 \to M \to 0$ is called a projective resolution of M and the sequence $0 \to N \to Q_0 \to Q_1 \to \cdots$ is called an injective resolution of N. Consider the commutative diagram of abelian groups

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \operatorname{Hom}_R(P_0, N) & \longrightarrow & \operatorname{Hom}_R(P_1, N) & \longrightarrow & \operatorname{Hom}_R(P_2, N) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_R(M, Q_0) & \longrightarrow & \operatorname{Hom}_R(P_0, Q_0) & \longrightarrow & \operatorname{Hom}_R(P_1, Q_0) & \longrightarrow & \operatorname{Hom}_R(P_2, Q_0) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_R(M, Q_1) & \longrightarrow & \operatorname{Hom}_R(P_0, Q_1) & \longrightarrow & \operatorname{Hom}_R(P_1, Q_1) & \longrightarrow & \operatorname{Hom}_R(P_2, Q_1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_R(M, Q_2) & \longrightarrow & \operatorname{Hom}_R(P_0, Q_2) & \longrightarrow & \operatorname{Hom}_R(P_1, Q_2) & \longrightarrow & \operatorname{Hom}_R(P_2, Q_2) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

(a) Show that the first row and first column are cochain complexes.

(b) Show that all other rows and columns are exact.

(c) Use diagram chasing to obtain isomorphisms

$$\frac{\ker(\mathrm{Hom}_R(P_k,N)\to \mathrm{Hom}_R(P_{k+1},N))}{\mathrm{im}(\mathrm{Hom}_R(P_{k-1},N)\to \mathrm{Hom}_R(P_k,N))}\cong \frac{\ker(\mathrm{Hom}_R(M,Q_k)\to \mathrm{Hom}_R(M,Q_{k+1}))}{\mathrm{im}(\mathrm{Hom}_R(M,Q_{k-1})\to \mathrm{Hom}_R(M,Q_k))}$$

for each $k \geq 0$ where we set $P_{-1}=0$ and $Q_{-1}=0$.

(d) Assume that every R-module has a projective resolution and an injective resolution. Show that the above isomorphism defines an abelian group $\mathrm{Ext}^k_R(M,N)$ that does not depend on the choice of the projective resolution of M or the injective resolution of N.

(e) Show that $\mathrm{Ext}^0_R(M,N)\cong \mathrm{Hom}_R(M,N)$.

(f) Show that $M\to M'$ induces a homomorphism of abelian groups $\mathrm{Ext}^k_R(M',N)\to \mathrm{Ext}^k_R(M,N)$.

(g) Show that $N\to N'$ induces a homomorphism of abelian groups $\mathrm{Ext}^k_R(M,N)\to \mathrm{Ext}^k_R(M,N')$.

(h) Show that a short exact sequence of R-modules $0\to M\to M'\to M''\to 0$ induces a long exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(M'',N) \longrightarrow \mathrm{Hom}_R(M',N) \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Ext}^1_R(M'',N) \longrightarrow \cdots$$

(i) Show that a short exact sequence of R-modules $0\to N\to N'\to N''\to 0$ induces a long exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,N') \longrightarrow \mathrm{Hom}_R(M,N'') \longrightarrow \mathrm{Ext}^1_R(M,N) \longrightarrow \cdots$$

Tricky Problems

1. Suppose that R is an integral domain. Let a and b be positive integers with $\gcd(a,b)=1$. Consider the ring homomorphism $\varphi\colon R[x,y]\to R[t]$ defined by $\varphi(x)=t^b$ and $\varphi(y)=t^a$.

   (a) Show that $(x^a-y^b)\subseteq \ker\varphi$.

   (b) Let $f(x,y)\in\ker\varphi$. Show that we can write $f(x,y)=g(x,y)+h(x,y)$ with $g(x,y)\in(x^a-y^b)$ and $\deg_y h(x,y)\leq b-1$.

   (c) Show that $h(x,y)\in\ker\varphi$.

   (d) Show that the exponents of $\varphi(x^iy^j)$ are distinct for $0\leq j\leq b-1$ and deduce that $h(x,y)=0$.

   (e) Show that $\ker\varphi=(x^a-y^b)$.

   (f) Show that (x^a-y^b) is a prime ideal of $R[x,y]$.

2. The Fibonacci sequence is defined by $F_0=0$, $F_1=1$, and $F_{n+1}=F_n+F_{n-1}$. Define the Fibonacci sequence in K by $F^K_0=0$, $F^K_1=1$, and $F^K_{n+1}=F^K_n+F^K_{n-1}$ for all $n\geq 1$. Consider the polynomial $p(x)=x^2-x-1$ in $K[x]$.

   (a) Show that if φ and ψ are distinct roots of $p(x)$ then $F^K_n=(\varphi^n-\psi^n)/(\varphi-\psi)$ for all $n\geq 0$.

   (b) Show that if φ and ψ are distinct roots of $p(x)$ then $F^K_n=0$ if and only if $\varphi^n=\psi^n$.

   (c) Let $p\neq 2,5$ be a prime and set $K=\mathbb{Z}/p\mathbb{Z}$. Show that if K has a square root of 5 then $F^K_{p-1}=0$.

   (d) Show that if $\left(\frac{5}{p}\right)=1$ then F_{p-1} is divisible by p.

   (e) Use the law of quadratic reciprocity to show that if $p\equiv 1,4 \pmod 5$ then F_{p-1} is divisible by p.

   (f) Make an interesting conjecture in the case where $p\equiv 2,3 \pmod 5$.