Tetration

Thomas Browning

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1 Tetration Analysis

Let $a > 0$, let $b_0 = 0$, and let $b_{n+1} = a^{b_n}$ for all integers $n \geq 0$. The purpose of this note is to determine the limiting behavior of the sequence $\{b_n\}$.

1.1 Case I: $a > 1$

We will rely on the fact that $x \mapsto a^x$ is strictly increasing (i.e., if $x < y$ then $a^x < a^y$).

(1) Observe that $b_0 = 0 < a = b_1$, and if $b_n < b_{n+1}$ then $b_{n+1} = a^{b_n} < a^{b_{n+1}} = b_{n+2}$. Then induction on n shows that the sequence $\{b_n\}$ is increasing.

(2) Suppose that $b_n \to L$. Then $b_{n+1} = a^{b_n} \to a^L$ by continuity of $x \mapsto a^x$. However, the shifted sequence $\{b_{n+1}\}$ must converge to the same limit as the original sequence $\{b_n\}$. This forces $a^L = L$.

(3) Suppose that $a^L = L$. Observe that $b_0 = 0 < a^L = L$, and if $b_n < L$ then $b_{n+1} = a^{b_n} < a^L = L$. Then induction on n shows that the sequence $\{b_n\}$ is bounded above by L.

We can combine these three results to completely determine the behaviour of the sequence $\{b_n\}$.

- If there are no solutions to $a^L = L$ then the sequence $\{b_n\}$ diverges.

  Proof. This follows from (2). □

- If there is a solution to $a^x = x$, then the sequence $\{b_n\}$ converges to the smallest solution to $a^x = x$.

  Proof. Suppose that there is a solution to $a^x = x$. By (3), $\{b_n\}$ is bounded above. By (1) and the monotone convergence theorem, $\{b_n\}$ converges. Let $b_n \to L$. By (2), L satisfies $a^L = L$. Suppose that L' also satisfies $a^{L'} = L'$. By (3), $\{b_n\}$ is bounded above by L'. Since $b_n \to L$, we must have $L \leq L'$. This shows that L is the smallest solution to $a^x = x$. □

Example 1. Let $a = \sqrt{2}$. The equation $a^x = x$ has two solutions: $x = 2$ and $x = 4$. Thus, $b_n \to 2$.

1.2 Case II: $a < 1$

We will rely on the fact that $x \mapsto a^x$ is strictly decreasing (i.e., if $x < y$ then $a^x > a^y$). Starting with $0 < a < 1$ and repeatedly applying this inclusion-reversing property gives the following sequence of inequalities:

$$\begin{gathered} b_0 < b_2 < b_1 \\ b_1 > b_3 > b_2 \\ b_2 < b_4 < b_3 \\ b_3 > b_5 > b_4 \\ \vdots \end{gathered}$$

Putting these together gives the ordering

$$b_0 < b_2 < b_4 < \cdots < b_5 < b_3 < b_1.$$

By the monotone convergence theorem, the subsequences $\{b_{2n}\}$ and $\{b_{2n+1}\}$ both converge. The same techniques as in the previous section allow us to determine the limits of these two sequences.

- The sequence $\{b_{2n}\}$ converges to the smallest solution to $a^{a^x} = x$.

  Proof. Let $b_{2n} \to L$. Then $b_{2(n+1)} = a^{a^{b_{2n}}} \to a^{a^L}$ by continuity of $x \mapsto a^{a^x}$. However, the shifted sequence $\{b_{2(n+1)}\}$ must converge to the same limit as the original sequence $\{b_{2n}\}$. This forces $a^{a^L} = L$. Suppose that L' also satisfies $a^{a^{L'}} = L'$. Observe that $b_0 = 0 < a^{a^{L'}} = L'$, and if $b_{2n} < L'$ then $b_{2(n+1)} = a^{a^{b_{2n}}} < a^{a^{L'}} = L'$. Then induction on n shows that the sequence $\{b_{2n}\}$ is bounded above by L'. Since $b_{2n} \to L$, we must have $L \le L'$. This shows that L is the smallest solution to $a^{a^x} = x$. $\square$

- The sequence $\{b_{2n+1}\}$ converges to the largest solution to $a^{a^x} = x$.

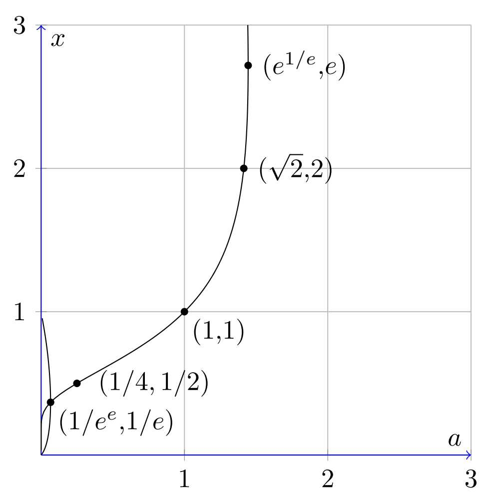
  Proof. Let $b_{2n+1} \to L$. Then $b_{2(n+1)+1} = a^{a^{b_{2n+1}}} \to a^{a^L}$ by continuity of $x \mapsto a^{a^x}$. However, the shifted sequence $\{b_{2(n+1)+1}\}$ must converge to the same limit as the original sequence $\{b_{2n+1}\}$. This forces $a^{a^L} = L$. Suppose that L' also satisfies $a^{a^{L'}} = L'$. Observe that $b_1 = a^0 > a^{a^{L'}} = L'$, and if $b_{2n+1} > L'$ then $b_{2(n+1)+1} = a^{a^{b_{2n+1}}} > a^{a^{L'}} = L'$. Then induction on n shows that the sequence $\{b_{2n+1}\}$ is bounded below by L'. Since $b_{2n+1} \to L$, we must have $L \ge L'$. This shows that L is the largest solution to $a^{a^x} = x$. $\square$

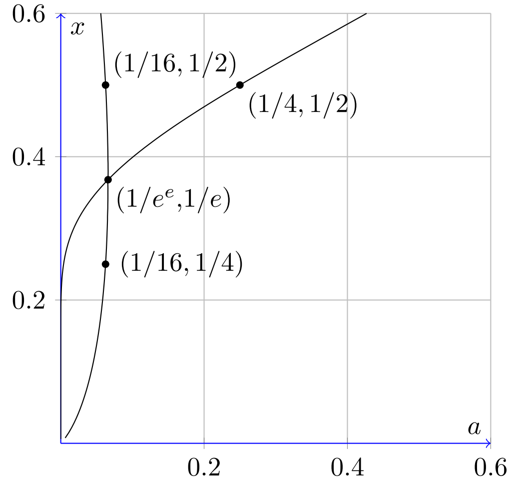
Example 2. Let $a = 1/4$. The equation $a^{a^x} = x$ has one solution: $x = 1/2$. Thus, $b_n \to 1/2$.

Example 3. Let $a = 1/16$. The equation $a^{a^x} = x$ has three solutions: $x = 1/4$, $x = 1/2$, and $x \approx 0.36425$. Thus, $b_{2n} \to 1/4$ and $b_{2n+1} \to 1/2$.

1.3 A Graph

We now study the solutions to the equation $a^{a^x} = x$. The graph has two components. The first component is the unbounded curve $a^x = x$. The second component is the curve between $(0, 0)$ and $(0, 1)$.





From the graphs, we obtain the following observations:

1. If $a > e^{1/e}$ then the equation $a^{a^x} = x$ has no solutions.
2. If $1/e^e \leq a \leq e^{1/e}$ then the equation $a^{a^x} = x$ has one solution, and it is on the first component $a^x = x$.
3. If $0 < a < 1/e^e$ then the equation $a^{a^x} = x$ has three solutions. The middle solution is on the first component $a^x = x$, but the smallest and largest solutions are on the second component.

Challenge: Prove these three observations. The parametrizations in the next section might be helpful.

Combining these three observations with the previous analysis proves the following theorem.

Theorem 1. *If $a > e^{1/e}$ then $\{b_n\}$ tends to infinity. If $1/e^e \leq a \leq e^{1/e}$ then $\{b_n\}$ converges. If $0 < a < 1/e^e$ then $\{b_n\}$ does not converge, but $\{b_{2n}\}$ and $\{b_{2n+1}\}$ both converge.*

1.4 Parametrization

The first component $a^x = x$ has the parametrization

$$(a, x) = \left(t^{1/t}, t\right), \qquad 0 < t < \infty.$$

To find a parametrization of the second component, consider the equation $a^{a^x} = x$. Raising both sides to the xth power gives the equivalent equation $(a^x)^{(a^x)} = x^x$. Let $t = a^x/x$. Then $a^x = tx$ so $(tx)^{(tx)} = x^x$. Taking xth roots of both sides gives $(tx)^t = x$. Then $x = t^{t/(1-t)}$ and $a = (tx)^{1/x}$. This gives the parametrization

$$(a, x) = \left(\left(t^{\left(t^{-t/(1-t)}\right)/(1-t)}\right), t^{t/(1-t)}\right), \qquad 0 < t < \infty.$$