

1 Tetration Analysis

Let $a > 0$, let $b_0 = 0$, and let $b_{n+1} = a^{b_n}$ for all integers $n \geq 0$. The purpose of this note is to determine the limiting behavior of the sequence $\{b_n\}$.

1.1 Case I: $a > 1$

We will rely on the fact that $x \mapsto a^x$ is strictly increasing (i.e., if $x < y$ then $a^x < a^y$).

1. Observe that $b_0 = 0 < a = b_1$, and if $b_n < b_{n+1}$ then $b_{n+1} = a^{b_n} < a^{b_{n+1}} = b_{n+2}$. Then induction on $n$ shows that the sequence $\{b_n\}$ is increasing.

2. Suppose that $b_n \to L$. Then $b_{n+1} = a^{b_n} \to a^L$ by continuity of $x \mapsto a^x$. However, the shifted sequence $\{b_{n+1}\}$ must converge to the same limit as the original sequence $\{b_n\}$. This forces $a^L = L$.

3. Suppose that $a^L = L$. Observe that $b_0 = 0 < a^L = L$, and if $b_n < L$ then $b_{n+1} = a^{b_n} < a^L = L$. Then induction on $n$ shows that the sequence $\{b_n\}$ is bounded above by $L$.

We can combine these three results to completely determine the behaviour of the sequence $\{b_n\}$.

- If there are no solutions to $a^L = L$ then the sequence $\{b_n\}$ diverges.

Proof. This follows from (2).

- If there is a solution to $a^x = x$, then the sequence $\{b_n\}$ converges to the smallest solution to $a^x = x$.

Proof. Suppose that there is a solution to $a^x = x$. By (3), $\{b_n\}$ is bounded above. By (1) and the monotone convergence theorem, $\{b_n\}$ converges. Let $b_n \to L$. By (2), $L$ satisfies $a^L = L$. Suppose that $L'$ also satisfies $a^{L'} = L'$. By (3), $\{b_n\}$ is bounded above by $L'$. Since $b_n \to L$, we must have $L \leq L'$.

This shows that $L$ is the smallest solution to $a^x = x$.

Example 1. Let $a = \sqrt{2}$. The equation $a^x = x$ has two solutions: $x = 2$ and $x = 4$. Thus, $b_n \to 2$.

1.2 Case II: $a < 1$

We will rely on the fact that $x \mapsto a^x$ is strictly decreasing (i.e., if $x < y$ then $a^x > a^y$). Starting with $0 < a < 1$ and repeatedly applying this inclusion-reversing property gives the following sequence of inequalities:

\begin{align*}
    b_0 &< b_2 < b_1 \\
    b_1 &> b_3 > b_2 \\
    b_2 &< b_4 < b_3 \\
    b_3 &> b_5 > b_4 \\
    &\vdots
\end{align*}
Putting these together gives the ordering

\[ b_0 < b_2 < b_4 < \cdots < b_5 < b_3 < b_1. \]

By the monotone convergence theorem, the subsequences \{b_{2n}\} and \{b_{2n+1}\} both converge. The same techniques as in the previous section allow us to determine the limits of these two sequences.

- The sequence \{b_{2n}\} converges to the smallest solution to \(a^{x^2} = x\).

Proof. Let \(b_{2n} \to L\). Then \(b_{2(n+1)} = a^{b_{2n}} \to a^{L^2}\) by continuity of \(x \mapsto a^{x^2}\). However, the shifted sequence \(\{b_{2(n+1)}\}\) must converge to the same limit as the original sequence \(\{b_{2n}\}\). This forces \(a^{L^2} = L\).

Suppose that \(L'\) also satisfies \(a^{L'^2} = L'\). Observe that \(b_0 = 0 < a^{L'^2} = L'\), and if \(b_{2n} < L'\) then \(b_{2(n+1)} = a^{b_{2n}} < a^{L'^2} = L'\). Then induction on \(n\) shows that the sequence \(\{b_{2n}\}\) is bounded above by \(L'\). Since \(b_{2n} \to L\), we must have \(L \leq L'\). This shows that \(L\) is the smallest solution to \(a^{x^2} = x\). \(\square\)

- The sequence \(\{b_{2n+1}\}\) converges to the largest solution to \(a^{x^2} = x\).

Proof. Let \(b_{2n+1} \to L\). Then \(b_{2(n+1)+1} = a^{b_{2n+1}} \to a^{L^2}\) by continuity of \(x \mapsto a^{x^2}\). However, the shifted sequence \(\{b_{2(n+1)+1}\}\) must converge to the same limit as the original sequence \(\{b_{2n+1}\}\). This forces \(a^{L^2} = L\). Suppose that \(L'\) also satisfies \(a^{L'^2} = L'\). Observe that \(b_1 = a^0 > a^{L'^2} = L'\), and if \(b_{2n+1} > L'\) then \(b_{2(n+1)+1} = a^{b_{2n+1}} > a^{L'^2} = L'\). Then induction on \(n\) shows that the sequence \(\{b_{2n+1}\}\) is bounded below by \(L'\). Since \(b_{2n+1} \to L\), we must have \(L \geq L'\). This shows that \(L\) is the largest solution to \(a^{x^2} = x\). \(\square\)

Example 2. Let \(a = 1/4\). The equation \(a^{x^2} = x\) has one solution: \(x = 1/2\). Thus, \(b_n \to 1/2\).

Example 3. Let \(a = 1/16\). The equation \(a^{x^2} = x\) has three solutions: \(x = 1/4, x = 1/2,\) and \(x \approx 0.36425\). Thus, \(b_{2n} \to 1/4\) and \(b_{2n+1} \to 1/2\).

1.3 A Graph

We now study the solutions to the equation \(a^{x^2} = x\). The graph has two components. The first component is the unbounded curve \(a^x = x\). The second component is the curve between \((0, 0)\) and \((0, 1)\).
From the graphs, we obtain the following observations:

1. If \( a > e^{1/e} \) then the equation \( a^x = x \) has no solutions.

2. If \( 1/e^e \leq a \leq e^{1/e} \) then the equation \( a^x = x \) has one solution, and it is on the first component \( a^x = x \).

3. If \( 0 < a < 1/e^e \) then the equation \( a^x = x \) has three solutions. The middle solution is on the first component \( a^x = x \), but the smallest and largest solutions are on the second component.

**Challenge:** Prove these three observations. The parametrizations in the next section might be helpful.

Combining these three observations with the previous analysis proves the following theorem.

**Theorem 1.** If \( a > e^{1/e} \) then \( \{b_n\} \) tends to infinity. If \( 1/e^e \leq a \leq e^{1/e} \) then \( \{b_n\} \) converges. If \( 0 < a < 1/e^e \) then \( \{b_n\} \) does not converge, but \( \{b_{2n}\} \) and \( \{b_{2n+1}\} \) both converge.

### 1.4 Parametrization

The first component \( a^x = x \) has the parametrization

\[
(a, x) = \left( t^{1/t}, t \right), \quad 0 < t < \infty.
\]

To find a parametrization of the second component, consider the equation \( a^x = x \). Raising both sides to the \( x \)th power gives the equivalent equation \( (a^x)^x = x^x \). Let \( t = a^x/x \). Then \( a^x = tx \) so \( (tx)^{(tx)} = x^x \). Taking \( x \)th roots of both sides gives \( (tx)^x = x \). Then \( x = t^{x/(1-t)} \) and \( a = (tx)^{1/x} \). This gives the parametrization

\[
(a, x) = \left( \left(t^{(t^{x/(1-t)}/(1-t)}/(1-t)}\right)^{(t^{x/(1-t)})/(1-t)}, t^{x/(1-t)} \right), \quad 0 < t < \infty.
\]