Projective Planes

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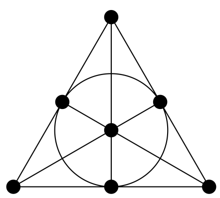
1 Transit Systems

A **transit system** consists of a finite set $\mathcal{P}$ of **bus stops**, a finite set $\mathcal{L}$ of **bus routes**, and an incidence relation $\sim \subseteq \mathcal{P} \times \mathcal{L}$.

Definition 1. A transit system is **efficient** if there is a bus route through any two bus stops.

Definition 2. A transit system is **economical** if there is at most one bus route through any two bus stops.

Example 3. The **Fano plane** is an efficient economical transit system with 7 bus route and 7 bus lines.



There are two degenerate situations that we will want to exclude.

Definition 4. A transit system is **degenerate** if there is a bus route that passes through every bus stop or a bus stop that is on every bus route.

Theorem 5. *A non-degenerate economical efficient transit system must have* $|\mathcal{L}| \geq |\mathcal{P}|$.

Proof. Assume that $|\mathcal{L}| \leq |\mathcal{P}|$. We will show that $|\mathcal{L}| = |\mathcal{P}|$. For each bus route $\ell \in \mathcal{L}$, let n_ℓ denote the number of bus stops on ℓ. For each bus stop $p \in \mathcal{P}$, let n_p denote the number of bus routes through p. The first observation is that if $p \not\sim \ell$, then $n_\ell \leq n_p$ by efficiency and economy. But $\sum_\ell n_\ell = \sum_p n_p$. If we can produce an injective function $f\colon \mathcal{L} \to \mathcal{P}$ satisfying $f(\ell) \not\sim \ell$, then we will have

$$\sum_\ell n_\ell \leq \sum_\ell n_{f(\ell)} \leq \sum_p n_p$$

with equality if and only if f is bijective satisfying $n_\ell = n_{f(\ell)}$. If we view $f\colon \mathcal{L} \to \mathcal{P}$ as a graph-theoretic matching, then Hall's marriage theorem states that such a matching exists if and only if for every subset $S \subseteq \mathcal{L}$, the neighborhood

$$N(S) = \{p \in \mathcal{P} : p \text{ is not a common bus stop of all of the bus routes } \ell \in S\}$$

satisfies $|S| \leq |N(S)|$. We can manually check that this condition holds for all subsets $S \subseteq \mathcal{L}$.

- If $S = \varnothing$, then $N(S) = \varnothing$.
- If $S = \{\ell\}$, then $|N(S)| = |\mathcal{P}| - n_\ell \geq 1$ by non-degeneracy.

- If $|S| \geq 2$, then $|N(S)| \geq |\mathcal{P}| - 1$ by economy.
- If $S = \mathcal{L}$, then $N(S) = \mathcal{P}$ by non-degeneracy.

Then our original assumption that $|\mathcal{L}| \leq |\mathcal{P}|$ ensures that $|S| \leq |N(S)|$ for every subset $S \subseteq \mathcal{L}$. □

Theorem 6. *If a non-degenerate economical efficient transit system has $|\mathcal{L}| = |\mathcal{P}|$, then any two bus routes share a unique stop.*

Proof. From the proof of the previous theorem, we must have $n_\ell = n_{f(\ell)}$ for all bus routes $l \in \mathcal{L}$. Then

$$\sum_{p \sim \ell} n_\ell = \sum_\ell n_\ell^2 = \sum_\ell n_{f(\ell)}^2 = \sum_p n_p^2 = \sum_{p \sim \ell} n_p,$$

but

$$\sum_{p,\ell} n_\ell = |\mathcal{P}| \sum_\ell n_\ell = |\mathcal{L}| \sum_p n_p = \sum_{p,\ell} n_p,$$

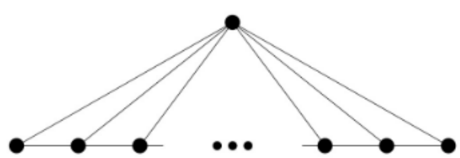
so we must have

$$\sum_{p \not\sim \ell} n_\ell = \sum_{p \not\sim \ell} n_p.$$

But $p \not\sim \ell$ implies $n_\ell \leq n_p$. Thus, we must have $n_\ell = n_p$ whenever $p \not\sim \ell$.

Returning to the statement of the theorem, it is enough to show that any two distinct bus routes $\ell, \ell' \in \mathcal{L}$ intersect, since uniqueness will follow from economy. First note that $|\mathcal{P}| = |\mathcal{L}| \geq 2$. By efficiency, we must have $n_p \geq 1$ for all bus stops $p \in \mathcal{P}$. By non-degeneracy, there exists a bus stop $p \not\sim \ell$. Then $n_\ell = n_p \geq 1$. Then there exists a bus stop $p' \sim \ell$. If $p' \sim \ell'$, then we are done. If $p' \not\sim \ell'$, then $n_{p'} = n_{\ell'}$. As in the proof of the previous theorem, efficiency and economy give an injective function from bus stops on ℓ' to bus routes through p'. But $n_{p'} = n_{\ell'}$, so this injective function must be surjective. In particular, the bus route ℓ through p' must pass through a bus stop on ℓ', as desired. □

Excluding one last slightly degenerate case (below) gives us the notion of a **projective plane**.



A projective plane has an **order** $q > 1$ such that $n_p = q + 1$ and $n_\ell = q + 1$. Then $|\mathcal{P}| = |\mathcal{L}| = 1 + q + q^2$ since each of the $q + 1$ bus routes through a fixed bus stop contributes an additional q bus stops.

If q is a prime power, then one way to construct a finite projective plane of order q is to consider the subspaces of $\mathbb{F}_q^3$ of dimensions 1 and 2. These are the Pappian/Desarguesian planes (technically Pappian planes are those over fields, and Desarguesian planes are those over division rings, but Wedderburn's little theorem states that a finite division ring is a field). In particular, there is a unique Pappian/Desarguesian plane of each prime power order. Are there others? (OEIS sequence A001231)

q	2	3	4	5	6	7	8	9	10	11	12
#	1	1	1	1	0	1	1	4	0	1?	0?

The Bruck-Ryser-Chowla theorem statues that if $q \equiv 1, 2 \pmod 4$ is not a sum of two squares (e.g., $q = 6$), then there is no projective plane of order q.

Conjecture 7. *Every finite projective plane has prime power order. Every finite projective plane of prime order is Pappian/Desarguesian.*