Odlyzko Bounds

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This is an exposition of Poitou's paper *Sur les petites discriminants.*

1 Minkwoski Bounds

Let K be a number field with signature $r_1+2r_2=n$. Minkowski's bound says that every ideal class contains an ideal of norm at most

$$\sqrt{|\Delta_K|}\left(\frac{4}{\pi}\right)^{r_2}\frac{n!}{n^n}.$$

This quantity must be at least one, so solving for the root discriminant $|\Delta_K|^{1/n}$ gives the lower bound

$$|\Delta_K|^{1/n}\geq\left(\frac{\pi}{4}\right)^{2r_2/n}\left(\frac{n^n}{n!}\right)^{2/n}\approx\left(e^2\right)^{r_1/n}\left(\frac{e^2\pi}{4}\right)^{2r_2/n}=(7.3890\ldots)^{r_1/n}(5.8033\ldots)^{2r_2/n}.$$

This lower bound increases with n, but is asymptotically constant for large n, with the exact limiting constant depending on the signature of K.

Example 1. The Hilbert class field H_K has the same root discriminant as K. If the root discriminant of K is below the limiting constant, then this gives a bound on the degree of H_K, and hence a bound on the class number of K. For example, $\mathbb{Q}(\zeta_5)$ has root discriminant $5^{3/4}\approx 3.34$. If $n\geq 8$, then $|\Delta_K|^{1/n}\geq 3.54$. Thus, $\mathbb{Q}(\zeta_5)$ has class number 1. But if the root discriminant of K is above the limiting constant, then we cannot conclude anything about the class number of K. For example, $\mathbb{Q}(\zeta_{11})$ has root discriminant $11^{9/10}\approx 8.65$.

The Odlyzko bounds are the improved asymptotic lower bounds

$$|\Delta_K|^{1/n}\gtrsim(4\pi e^{1+\gamma})^{r_1/n}(4\pi e^{\gamma})^{2r_2/n}=(60.8395\ldots)^{r_1/n}(22.3816\ldots)^{2r_2/n}$$

and with GRH

$$|\Delta_K|^{1/n}\gtrsim(8\pi e^{\gamma+\pi/2})^{r_1/n}(8\pi e^{\gamma})^{2r_2/n}=(215.3325\ldots)^{r_1/n}(44.7632\ldots)^{2r_2/n}.$$

Remark 2. The existence of infinite class field towers places a limit on how much these constants can be improved. For example, the class field tower of $\mathbb{Q}(\zeta_{11}+\zeta_{11}^{-1},\sqrt{-46})$ is infinite, so there are arbitrary large number fields with the same root discriminant ≈ 92.4.

In this writeup, we will see how Odlyzko's constants arise and how the Odlyzko bounds can be made explicit for small n. One notable appearance of Poitou's explicit Odlyzko bounds is in the Buzzard-Taylor Lean proof of Fermat's Last Theorem where they are used to bound the fixed field of the kernel of a Galois representation with limited ramification.

2 Contour Integration

The Dedekind zeta function

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \|\mathfrak{p}\|^{-s}}$$

has completion

$$\Lambda_K(s) = |\Delta_K|^{s/2}(\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{-s}\Gamma(s))^{r_2}\zeta_K(s)$$

and functional equation $\Lambda_K(s) = \Lambda_K(1-s)$. Let R_T denote the boundary of the rectangle $[-\epsilon, 1+\epsilon]\times[-T, T]$. We will integrate the logarithmic derivative $\Lambda_K'(s)/\Lambda_K(s)$ against a holomorphic function $\Phi(s)$, giving

$$\frac{1}{2\pi i}\int_{R_T} \frac{\Lambda_K'(s)}{\Lambda_K(s)}\,\Phi(s)\,ds = -\Phi(0) - \Phi(1) + \sum_{|\mathrm{Im}\,\rho|<T} \Phi(\rho)$$

where the sum runs over the nontrivial zeros ρ of $\zeta_K(s)$. We will set

$$\Phi(s) = \int_{-\infty}^{\infty} F(x)\cosh((s-\tfrac{1}{2})x)\,dx = \int_{-\infty}^{\infty} F(x)\exp((s-\tfrac{1}{2})x)\,dx$$

where $F\colon \mathbb{R}\to\mathbb{R}$ is an even function satisfying $F(0) = 1$. Then $\Phi(s) = \Phi(1-s)$, so the integrals over the vertical segments of R_T will be the same. And the decay $\Phi(\sigma+it)\to 0$ as $t\to\pm\infty$ will ensure that the integrals over the horizontal segments of R_T vanish as $T\to\infty$, resulting in the formula

$$\frac{1}{2\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} \frac{2\Lambda_K'(s)}{\Lambda_K(s)}\,\Phi(s)\,ds = -2\Phi(1) + \sum_{\rho}\Phi(\rho).$$

Now the logarithmic derivative $\Lambda_K'(s)/\Lambda_K(s)$ decomposes as a sum

$$\frac{\Lambda_K'(s)}{\Lambda_K(s)} = \frac{1}{2}\log|\Delta_K| + \frac{r_1}{2}(\psi(s/2) - \log\pi) + r_2(\psi(s) - \log 2\pi) - \sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{ms}}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the digamma function. Multiplying through by 2 and rearranging gives

$$\frac{2\Lambda_K'(s)}{\Lambda_K(s)} = \log|\Delta_K| + r_1(\psi(s/2) - \psi(s) + \log 2) + n(\psi(s) - \log 2\pi) - 2\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{ms}}.$$

The following table lists the contribution of each summand in terms of $F(x)$ and $f(x) = F(x)\cosh(x/2)$.

Summand of $\frac{2\Lambda_K'(s)}{\Lambda_K(s)}$	Contribution in terms of $F(x)$	Contribution in terms of $f(x)$
$\log\lvert\Delta_K\rvert$	$\log\lvert\Delta_K\rvert$	$\log\lvert\Delta_K\rvert$
$\psi(s/2) - \psi(s) + \log 2$	$-\frac{\pi}{2} + \int_0^{\infty}\frac{1-F(x)}{2\cosh(x/2)}\,dx$	$-1 + \int_0^{\infty}\frac{1-f(x)}{2\cosh^2(x/2)}\,dx$
$\psi(s) - \log 2\pi$	$-(\gamma + \log 8\pi) + \int_0^{\infty}\frac{1-F(x)}{2\sinh(x/2)}\,dx$	$-(\gamma + \log 4\pi) + \int_0^{\infty}\frac{1-f(x)}{\sinh(x)}\,dx$
$2\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\lVert\mathfrak{p}\rVert}{\lVert\mathfrak{p}\rVert^{ms}}$	$2\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\lVert\mathfrak{p}\rVert}{\lVert\mathfrak{p}\rVert^{m/2}}F(m\log\lVert\mathfrak{p}\rVert)$	$4\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\lVert\mathfrak{p}\rVert}{1+\lVert\mathfrak{p}\rVert^{m}}f(m\log\lVert\mathfrak{p}\rVert)$

Finally, we must turn our attention to the sum

$$\sum_\rho \Phi(\rho) = \operatorname{Re}\sum_\rho \Phi(\rho) = \sum_\rho \operatorname{Re}\Phi(\rho).$$

We can express $\operatorname{Re}\Phi(\sigma+it)$ as

$$\operatorname{Re}\Phi(\sigma+it) = \int_{-\infty}^{\infty} F(x)\cosh((\sigma-\tfrac{1}{2})x)\cos(tx)\,dx.$$

On the lines $\sigma = 0$ and $\sigma = 1$, nonnegativity of the Fourier transform of $f(x)$ would give

$$\operatorname{Re}\Phi(\sigma+it) = \int_{-\infty}^{\infty} f(x)\cos(tx)\,dx = \int_{-\infty}^{\infty} f(x)\exp(itx)\,dx \geq 0.$$

Since $\Phi(\sigma+it)\to 0$ as $t\to\pm\infty$, the maximum principle for harmonic functions would force $\operatorname{Re}\Phi(\sigma+it)\geq 0$ on the strip $0\leq\sigma\leq 1$. Whereas nonnegativity of the Fourier transform of $F(x)$ would immediately give

$$\operatorname{Re}\Phi(\tfrac{1}{2}+it) = \int_{-\infty}^{\infty} F(x)\cos(tx)\,dx = \int_{-\infty}^{\infty} F(x)\exp(itx)\,dx \geq 0.$$

Putting everything together, if $f(x)$ has nonnegative Fourier transform, then we have the lower bound

$$\begin{aligned}\log|\Delta_K| \geq n(\gamma+\log 4\pi) + r_1 - r_1\int_0^\infty \frac{1-f(x)}{2\cosh^2(x/2)}\,dx - n\int_0^\infty \frac{1-f(x)}{\sinh(x)}\,dx \\ + 4\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{1+\|\mathfrak{p}\|^m} f(m\log\|\mathfrak{p}\|) - 4\int_0^\infty f(x)\,dx.\end{aligned} \tag{1}$$

And with GRH, if $F(x)$ has nonnegative Fourier transform, then we have the lower bound

$$\begin{aligned}\log|\Delta_K| \geq n(\gamma+\log 8\pi) + r_1\frac{\pi}{2} - r_1\int_0^\infty \frac{1-F(x)}{2\cosh(x/2)}\,dx - n\int_0^\infty \frac{1-F(x)}{2\sinh(x/2)}\,dx \\ + 2\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{m/2}} F(m\log\|\mathfrak{p}\|) - 4\int_0^\infty F(x)\cosh(x/2)\,dx.\end{aligned} \tag{2}$$

Note that Odlyzko's constants are already present in these lower bounds.

3 Optimization

If $f(x)$ is nonnegative with nonnegative Fourier transform, then (1) gives the lower bound

$$\frac{1}{n}\log|\Delta_K| \geq \gamma+\log 4\pi + \frac{r_1}{n} - \int_0^\infty (1-f(xy))\,h(x)\,dx - \frac{4}{n}\int_0^\infty f(xy)\,dx,$$

where

$$h(x) = \frac{1}{\sinh(x)} + \frac{r_1}{n}\frac{1}{2\cosh^2(x/2)}.$$

We will set

$$f(x) = \left(\frac{3}{x^3}(\sin x - x\cos x)\right)^2.$$

This function arises as the square of the Fourier transform of the function

$$v(x) = \begin{cases} 1-x^2 & \text{if } |x|\leq 1, \\ 0 & \text{if } |x|\geq 1. \end{cases}$$

We should choose y to minimize the sum

$$\int_0^\infty (1 - f(xy))\, h(x)\, dx + \frac{4}{n} \int_0^\infty f(xy)\, dx = \frac{1}{y} \left(\frac{12\pi}{5n} + \int_0^\infty (1 - f(x))\, h(x/y)\, dx \right).$$

If K is totally imaginary, then the following table lists the optimal y and the corresponding lower bound.

n	y	$\lvert\Delta_K\rvert^{1/n}$
2	3.94	1.72
4	2.16	3.25
6	1.59	4.55
8	1.31	5.65
10	1.14	6.60
12	1.02	7.41
14	0.93	8.12
16	0.87	8.74
18	0.82	9.30
20	0.78	9.80

Example 3. Continuing from Example 1, we can now conclude that $\mathbb{Q}(\zeta_{11})$ has class number 1.

4 Appendix: Derivations

We now derive the formulas in the table at the bottom of page 2. We will need the integral representation

$$\psi(s) = \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-sx}}{1 - e^{-x}} \right) dx.$$

Let

$$\varphi(t) = \Phi(\tfrac{1}{2} + it) = \int_{-\infty}^\infty F(x) \cos(tx)\, dx = \int_{-\infty}^\infty F(x) \exp(itx)\, dx.$$

The Fourier inversion formula gives

$$\frac{1}{2\pi} \int_{-\infty}^\infty \varphi(t) \exp(-ity)\, dt = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty F(x) \exp(it(x-y))\, dx\, dt = F(y).$$

Then shifting the contour gives

$$\frac{1}{2\pi i} \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} \Phi(s)\, ds = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s)\, ds = \frac{1}{2\pi} \int_{-\infty}^\infty \varphi(t)\, dt = F(0) = 1,$$

and

$$\begin{aligned}
\frac{1}{2\pi i} \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} \Phi(s)\, \psi(s)\ ds &= \frac{1}{2\pi} \int_{-\infty}^\infty \varphi(t)\, \psi(\tfrac{1}{2} + it)\, ds \\
&= \psi(\tfrac{1}{2}) + \frac{1}{2\pi} \int_{-\infty}^\infty \varphi(t) \left[\psi(\tfrac{1}{2} + it) - \psi(\tfrac{1}{2}) \right] dt \\
&= \psi(\tfrac{1}{2}) + \frac{1}{2\pi} \int_{-\infty}^\infty \varphi(t) \int_0^\infty \frac{(1 - e^{-itx}) e^{-x/2}}{1 - e^{-x}}\, dx\, dt \\
&= \psi(\tfrac{1}{2}) + \int_0^\infty \frac{(1 - F(x)) e^{-x/2}}{1 - e^{-x}}\, dx \\
&= \psi(\tfrac{1}{2}) + \int_0^\infty \frac{1 - F(x)}{2 \sinh(x/2)}\, dx
\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty}\Phi(s)\,\psi(s/2)\,ds &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)\,\psi(\tfrac{1}{4}+i\tfrac{t}{2})\,dt\\ &= \psi(\tfrac{1}{4}) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)\left[\psi(\tfrac{1}{4}+i\tfrac{t}{2})-\psi(\tfrac{1}{4})\right]dt\\ &= \psi(\tfrac{1}{4}) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)\int_0^{\infty}\frac{(1-e^{-itx/2})e^{-x/4}}{1-e^{-x}}\,dx\,dt\\ &= \psi(\tfrac{1}{4}) + \int_0^{\infty}\frac{(1-F(x/2))e^{-x/4}}{1-e^{-x}}\,dx\\ &= \psi(\tfrac{1}{4}) + 2\int_0^{\infty}\frac{(1-F(x))e^{-x/2}}{1-e^{-2x}}\,dx,\end{aligned}$$

so

$$\begin{aligned}&\frac{1}{2\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty}\Phi(t)\left[\psi(s/2)-\psi(s)+\log 2\right]dt\\ &= \psi(\tfrac{1}{4})-\psi(\tfrac{1}{2})+\log 2+\int_0^{\infty}(1-F(x))e^{-x/2}\left(\frac{2}{1-e^{-2x}}-\frac{1}{1-e^{-x}}\right)dx\\ &= \psi(\tfrac{1}{4})-\psi(\tfrac{1}{2})+\log 2+\int_0^{\infty}\frac{(1-F(x))e^{-x/2}}{1+e^{-x}}\,dx\\ &= \psi(\tfrac{1}{4})-\psi(\tfrac{1}{2})+\log 2+\int_0^{\infty}\frac{1-F(x)}{2\cosh(x/2)}\,dx.\end{aligned}$$

Then the first three rows of the table follow from the special values $\psi(\frac{1}{2}) = -(\gamma+\log 4)$ and $\psi(\frac{1}{4}) = -(\frac{\pi}{2}+\gamma+\log 8)$. For the final row of the table, we cannot shift the contour, but we can directly compute

$$\begin{aligned}&\frac{1}{2\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty}\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{ms}}\,\Phi(s)\,ds\\ &\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{m(1+\epsilon+it)}}\,\Phi(1+\epsilon+it)\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{m(1+\epsilon+it)}}F(x)\exp((\tfrac{1}{2}+\epsilon+it)x)\,dx\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{m/2}}F(x)\exp((\tfrac{1}{2}+\epsilon+it)(x-m\log\|\mathfrak{p}\|))\,dx\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{m/2}}F(x+m\log\|\mathfrak{p}\|)\exp((\tfrac{1}{2}+\epsilon+it)x)\,dx\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{m/2}}F(x+m\log\|\mathfrak{p}\|)\exp((\tfrac{1}{2}+\epsilon)x)\exp(itx)\,dx\,dt\\ &= \sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log\|\mathfrak{p}\|}{\|\mathfrak{p}\|^{m/2}}F(m\log\|\mathfrak{p}\|)\end{aligned}$$

where the last step uses the Fourier inversion theorem.