Analysis of Musical Systems

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1 Introduction

In Western music, the octave is subdivided into twelve tones. This can be justified through equations such as $(3/2)^{12} \approx 2^7$ but these explanations do not provide an argument as to why a better musical system does not exist. Perhaps some other set of frequencies would result in a musical system better suited to the creation of music. This paper aims to mathematically and empirically approach the problem of finding the set of frequencies best suited for the construction of music.

2 Properties desirable of a Musical System

We shall use the term "musical system" to refer to a set of positive frequencies $\{f_i\}$ that functions well as a set of frequencies for musical notes to be drawn from. In particular, drawing frequencies of notes from the musical system should allow for the creation of sufficiently complex music that sounds good and is physically playable. For example, the (perfectly good) musical system that is used in Western music is $\{440 \cdot 2^{i/12}\}$ (In this paper, I will use the term "Western musical system" to refer to equal temperament). In this section, I shall provide conditions on the $\{f_i\}$ in order for the $\{f_i\}$ to function well as a musical system. In addition, I shall provide physical justification for why these conditions must hold. I claim that any well-functioning musical system $\{f_i\}$ must satify the following two conditions:

1. $\{f_i\}$ must be a two-sided geometric progression, say with common ratio r.
2. Every small positive integer must have a good approximations by a power of r.

To provide physical justification for this, I will first argue that for each f_i, there are not f_j arbitrarily close to f_i. In other words, there is an interval around each f_i containing no other f_j. If this were not the case then an infinite number of f_i would lie inside a finite interval which has several undesirable consequences. Firstly, we (or any arbitrarily sensitive device) would be unable to differentiate between some pairs of different notes in the musical system. Secondly, having an infinite number of notes in a finite interval would prevent sheet music in any form from being able to express every note in this finite interval. Thirdly, having an infinite number of notes in a finite interval isn't useful as no instrument can be played with arbitrarily fine precision. Thus, the frequencies in any well-functioning musical system must be isolated. In particular, any well-functioning musical system can be ordered as

$$\ldots \leq f_{-2} \leq f_{-1} \leq f_0 \leq f_1 \leq f_2 \leq \ldots$$

There are also several undesirable consequences if the $\{f_i\}$ were not equally spaced geometrically (so that f_{i+1}/f_i is not constant). Firstly, modulation in music requires that the musical system "looks the same" from the perspective of any note. In particular, modulating from one note to another should not affect the notes available relative to the new note. This would not be the case if the $\{f_i\}$ were not equally spaced geometrically. Secondly, if a piece of music is to be played by an instrument other than the intended instrument (which is common practice today) it may be necessary to transpose the music by some constant amount so that the resulting music lies in the range of the new instrument. In order to ensure that the transposed music sounds the same as the original, the $\{f_i\}$ must be equally spaced geometrically. Historically, the utility of equally spaced frequencies was realized around the time of Bach and is "argued for" by *The Well-Tempered Clavier*. Thus, the frequencies in any well-functioning musical system must be equally spaced which shows that the musical system must be of the form $\{ar^i\}$. For example, in Western music, $a = 440$ and $r = 2^{1/12}$. This shows the first condition.

As for the second condition, every sound that we hear with a definite pitch f has harmonics of frequency $2f$, $3f$, and so on. Thus, we are accustomed to hearing these intervals with small integer frequency ratio. In fact, intervals with integer frequency ratio 1 through 6 are consonant. Thus, every well-functioning musical system should have good approximations to these intervals which shows the second condition. Note that approximations should be measured by taking the logarithm of the frequency ratio between the approximation and the true value (effectively working in cents up to a constant factor). This can be mathematically expressed as

$$\text{For all small positive integers } n,\ \min_k |\ln(r^k/n)| \text{ must be small.}$$

By properties of logorithms, this is equivalent to

$$\text{For all small positive integers } n,\ \min_k |k \ln r - \ln n| \text{ must be small.}$$

Finally, $|k \ln r - \ln n|$ is minimized by $k = [\ln n / \ln r]$ where $[x]$ is x rounded to the nearest integer. Then

$$\text{For all small positive integers } n,\ \left| \left[\frac{\ln n}{\ln r} \right] \ln r - \ln n \right| \text{ must be small.}$$

Then to find values of r that work well, we can minimize the sum of squares of these deviations

$$f(r) = \sum_{n=2}^{N} \left(\left[\frac{\ln n}{\ln r} \right] \ln r - \ln n \right)^2.$$

Notice that the $n = 1$ term would contribute nothing because r^0 is a perfect approximation to $n = 1$. In Figure 1, there are plots (made in R) of $f(r)$ for $N = 5, 6$ and $1 \leq r \leq 1.3$ and $1 \leq r \leq 1.1$. Figure 2 is a table of significant minima of $f(r)$ for $N = 5, 6$. Figure 3 is a table of the errors of pure intervals in the musical systems resulting from the minima from Figure 2. Figure 4 is the same as Figure 3 but with the minima rounded to an nth-root of 2.

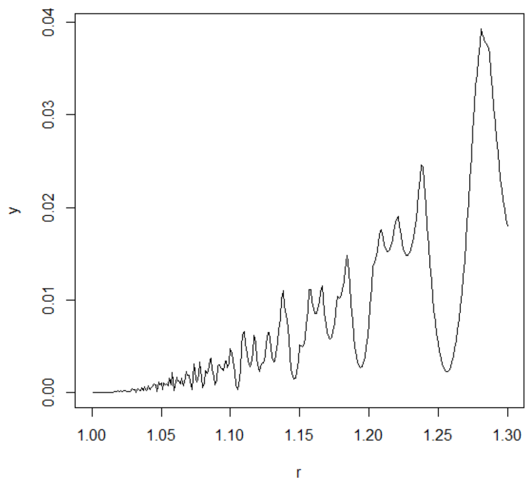
The similarity between the graphs for $N = 5$ and $N = 6$ in Figure 1 comes from the fact that $6 = 2 \times 3$. In particular, if there are good approximations to 2 and 3 by r^x and r^y respectively, then r^{x+y} will be a good approximation to 6. Thus, insisting that $N = 6$ also have a good approximation is almost redundant which is why the graphs for $N = 5$ and $N = 6$ are so similar in Figure 1. Additionally, in Figures 3 and 4, note that the errors for the perfect fourth and perfect fifth must sum up to the error for the octave and the errors for the minor third and major third must sum up to the error for the perfect fifth.

3 Analysis of the data

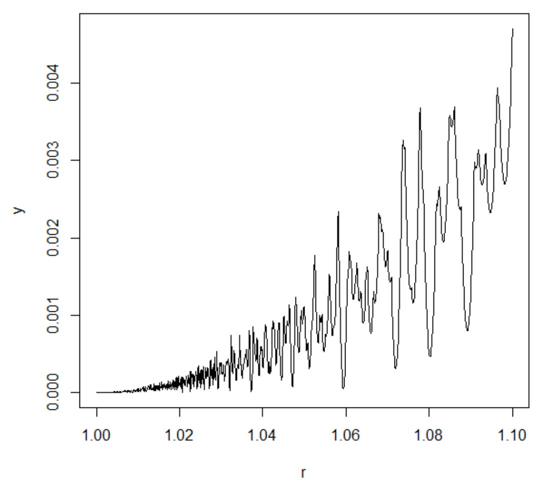
In the following analysis of the musical systems resulting from the figures, we will use the versions of the musical systems where the octave is pure (Figure 4 as opposed to Figure 3).

The first four values of r correspond to subdividing the octave into 3, 4, 5, and 7 intervals respectively. In these musical systems, the errors in the perfect fifth and perfect fourth are larger than 10 cents which is very noticeable. The fifth value of r is just the Western musical system of subdividing the octave into 12 semitones. Note that my equations suggest shrinking each semitone by 0.1 cents which minimizes the sums of squares despite giving the octave an error of 1.2 cents.

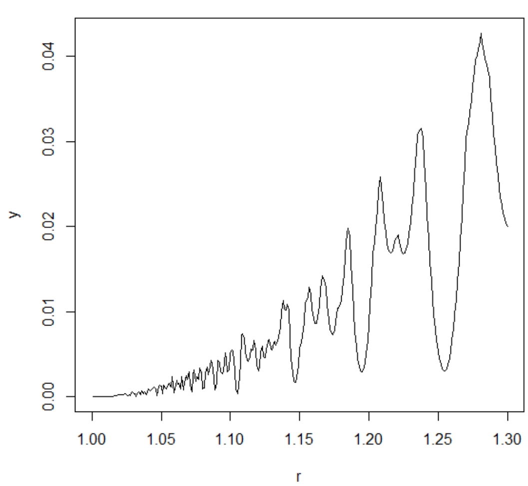
The remaining four values of r are much more interesting as they correspond to subdividing the octave into 19, 31, 34, and 51 intervals respectively but do so with much less error than in the Western musical



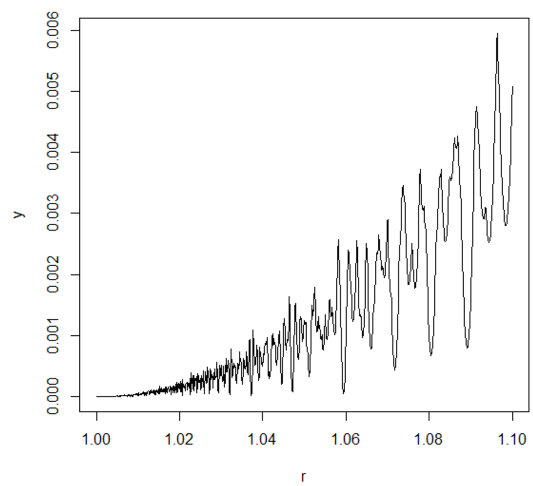
(a) $1 \leq r \leq 1.3$, $N = 5$



(b) $1 \leq r \leq 1.1$, $N = 5$



(c) $1 \leq r \leq 1.3$, $N = 6$



(d) $1 \leq r \leq 1.1$, $N = 6$

Figure 1: Plots of $f(r)$ for $N = 5$ and $N = 6$

$r \approx 2^{1/n}$	$f(r)$ $(N=5)$	$f(r)$ $(N=6)$
$1.25 \approx 2^{1/3}$	0.0023069	0.0030532
$1.19 \approx 2^{1/4}$	0.0026877	0.0029076
$1.146 \approx 2^{1/5}$	0.0014246	0.0016440
$1.105 \approx 2^{1/7}$	0.0003084	0.0003264
$1.059 \approx 2^{1/12}$	0.0000423	0.0000554
$1.037 \approx 2^{1/19}$	0.0000145	0.0000150
$1.023 \approx 2^{1/31}$	0.0000081	0.0000115
$1.021 \approx 2^{1/34}$	0.0000034	0.0000041
$1.013 \approx 2^{1/53}$	0.0000004	0.0000005

Figure 2: Significant minima of $f(r)$ for $N=5$ and $N=6$

r	Octave	Perfect Fifth	Perfect Fourth	Major Third	Minor Third
1.25448	−22 cents	+83 cents	−106 cents	+6 cents	+77 cents
1.19479	+32 cents	−85 cents	+118 cents	−78 cents	−8 cents
1.14671	−15 cents	+9 cents	−24 cents	+88 cents	−79 cents
1.10488	+9 cents	−11 cents	+20 cents	−41 cents	+30 cents
1.05940	−1.2 cents	−2.7 cents	+1.4 cents	+13.3 cents	−16 cents
1.03723	+2.38 cents	−5.84 cents	+8.22 cents	−6.62 cents	+0.77 cents
1.02263	+0.97 cents	−4.62 cents	5.59 cents	1.10 cents	−5.71 cents
1.02057	−1.49 cents	3.05 cents	−4.54 cents	1.44 cents	1.61 cents
1.01317	0.53 cents	0.24 cents	0.29 cents	−1.24 cents	1.48 cents

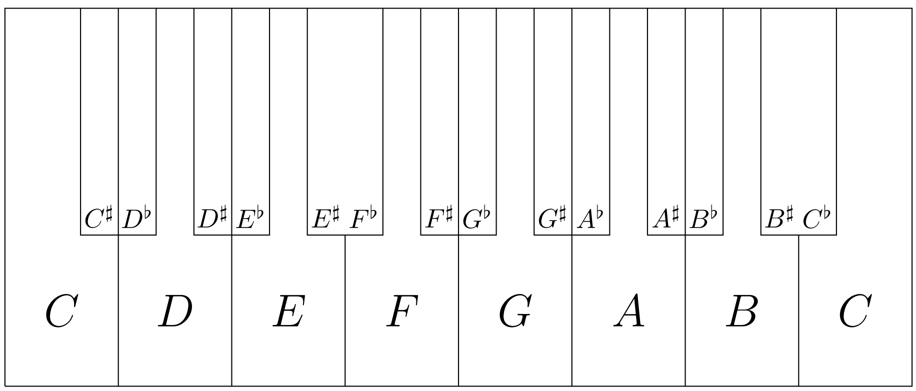
Figure 3: Error for r in Figure 2

r	Perfect Fifth	Perfect Fourth	Major Third	Minor Third
$2^{1/3}$	+98 cents	-98 cents	+13 cents	+83 cents
$2^{1/4}$	−102 cents	+102 cents	−86 cents	−16 cents
$2^{1/5}$	+18 cents	−18 cents	+94 cents	−76 cents
$2^{1/7}$	−16 cents	+16 cents	−43 cents	27 cents
$2^{1/12}$	−2.0 cents	+2.0 cents	+13.7 cents	−15.6 cents
$2^{1/19}$	−7.22 cents	+7.22 cents	−7.37 cents	+0.15 cents
$2^{1/31}$	−5.18 cents	+5.18 cents	+0.78 cents	−5.96 cents
$2^{1/34}$	+3.93 cents	−3.93 cents	+1.92 cents	2.01 cents
$2^{1/53}$	−0.07 cents	+0.07 cents	−1.41 cents	1.34 cents

Figure 4: Error when r rounded to nth-root of 2

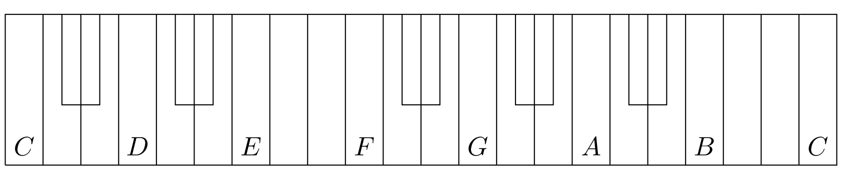
system. Compared to the Western musical system, the 19, 31, and 34 tone musical systems all improve on the 13-16 cent error in the major and minor thirds at the cost of increasing the 2 cent error in the perfect fourth and fifth by a couple cents. The 53 tone musical system, however, manages to get less than a 0.1 cent error on the perfect fifth and fourth and less than a 1.5 cent error on the major and minor thirds.

In the 19 tone musical system, the approximation to the minor third is 5 steps long so 4 minor thirds adds up to an octave plus one step. In other words, one step is roughly equal to the greater diesis. Note that one step has a frequency ratio of $2^{1/19} = 1.0372$ whereas the greater diesis has a frequency ratio of $(6/5)^4/2 = 1.0368$. The fact that the 19 tone musical system contains an approximation to the greater diesis explains why the error in the minor third is so small (only 0.15 cents). A disadvantage to the 19 tone musical system that J. Murray Barbour points out in [1] is that we are accustomed to hearing major thirds 13.7 cents sharp whereas major thirds in the 19 tone musical system are 7.4 cents flat. This 20 cent discrepancy makes the major thirds of the 19 tone musical system sound flatter than they actually are. A positive feature to the 19 tone musical system, however, is that our notation and our system of scales easily extend to it. In particular, the major third is 6 steps long and the minor third is 5 steps long which leads to the following 19 tone keyboard that appears in [1] and [2] (although in [2] the flat keys are raised up to a third row):



where the standard C-major scale is on the lower row of keys.

In the 31 tone musical system, the approximation to the major third is 10 steps long so 3 major thirds adds up to an octave minus one step. In other words, one step is roughly equal to the lesser diesis. Note that one step has a frequency ratio of $2^{1/31} = 1.0226$ whereas the lesser diesis has a frequency ratio of $2/(5/4)^3 = 1.024$. The fact that the 31 tone musical system contains an approximation to the lesser diesis explains why the error in the major third is so small (only 0.78 cents). Unlike the 19 tone musical system, the errors of the 31 tone musical system are entirely in the same directions as in the Western musical system which helps ease the transition from one to the other. Furthermore, just like the 19 tone musical system, the 31 tone musical system can adopt the standard system of notation and scales. In particular, the major third is 10 steps long and the minor third is 8 steps long. This leads to the following 31 tone keyboard that is mentioned in [1] and [2]:



The 34 tone musical system is much unlike the previous two. The step admits no simple description whereas the steps in the 19 and 31 tone musical systems were given by the greater and lesser diesis respectively. In addition, the 34 tone musical system does not easily adopt the standard system of notation and scales as [1] points out. In particular, if you construct a major scale where minor thirds are 9 steps long and major thirds are 11 steps long, the whole steps alternate between 5 and 6 steps long (but the two half steps are both 3 steps long). This makes it harder to extend the standard system of notation and scales to the 34 tone musical system than to the previous two systems.

Finally, the 53 tone musical system suffers from the same problems as the 34 tone musical system. The step admits no simple description and the 53 tone musical system does not easily adopt the standard system of notation and scales. In particular, if you construct a major sacle where minor thirds are 14 steps long and major thirds are 17 steps long, the whole steps alternate between 8 and 9 steps long (but the two half steps are both 5 steps long).

4 Conclusion

Firstly, the slight improvement in accuracy of the 34 tone musical system over the 31 tone musical system doesn't make up for the difficulty in extending standard notation and scales to the 34 tone musical system. Secondly, despite the accuracy of the 53 tone musical system, having 53 notes per octave is very impractical. For example, a piano has 88 keys so a piano tuned to the 53 tone musical system would have a range of less than 2 octaves. The problem would affect all instruments that can only play a finite number of notes. As for instruments like violin that can play in a continuous interval, it would be impossible to play accurately at any reasonable speed. This problem would also affect any musical system with more than 53 notes per octave which is why my tables stop at the 53 tone musical system. Then the only two musical systems that are still possibly practical (other than the Western system) are the 19 and 31 tone musical systems.

Depending on the instrument, the 31 tone musical system might still suffer from the problem of too many notes per octave. A piano could fit at around 3 octaves but instruments with a smaller range would have more trouble. The 19 tone musical system, however, doesn't suffer from this problem. In fact, with the keyboard layout for the 19 tone musical system shown above, the range of the piano wouldn't change.

In conclusion, the only musical systems that are practical to consider are the Western musical system, the 19 tone musical system, and the 31 tone musical system. The 19 tone musical system suffers from the fact that the major third differing from from the usual major third (of the Western musical system) by 20 cents. The 31 tone musical system may suffer from having too many notes per octave depending on the instrument. Both systems, however, largely correct the errors in the major third and minor third in the Western musical system. It is for this reason that (in my opinion), these musical systems are definitely worthy of more consideration and experiment than they have been receiving.

References

[1] J. Murray Barbour. *Tuning and Temperament.* Michigan State College Press, East Lansing, Michigan, 1951.

[2] Douglas Keislar. *History and Principles of Microtonal Keyboards.* Computer Music Journal, Vo. 11, No. 1, Microtonality (Spring 1987), pp. 18-28.