L^p EIGENFUNCTION BOUNDS FOR THE HERMITE OPERATOR

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ABSTRACT. We obtain L^p eigenfunction bounds for the harmonic oscillator $H = -\Delta + x^2$ in $\mathbb{R}^n$ and for other related operators, improving earlier results of Thangavelu and Karadzhov. We also construct suitable counterexamples which show that our estimates are sharp.

1. INTRODUCTION

The question of obtaining L^p eigenfunction bounds for elliptic operators on compact manifolds has been considered in Sogge's work, for which we refer the reader to his book [11]. The L^p eigenfunction bounds in [11] are sharp, and turn out to be related to a variable coefficient version of the restriction theorem, and further to a phase curvature condition for Fourier integral operators. In this analysis a special role is played by the Laplace-Beltrami operator for the sphere, which is the worst case because it has many highly concentrated eigenfunctions. This is connected to the fact that it has a periodic Hamilton flow.

In this article we consider the problem of obtaining L^p eigenfunction bounds for the Hermite operator $H = -\Delta + x^2$ in $\mathbb{R}^n$ and also for a larger class of related operators of the form $H_V = -\Delta + V$. Within this class the Hermite operator plays a role similar to that of the spherical Laplacian, in that it has a periodic Hamilton flow and many highly concentrated eigenfunctions.

This question has received considerable interest in the context of Riesz summability for the harmonic oscillator in the work of Thangavelu [12], [14], [13] and Karadzhov [7].

Our interest in it has a different source, namely the strong unique continuation problem for parabolic equations. In this context the work of Thangavelu and Karadzhov has already found applications in Escauriaza [2] and Escauriaza-Vega [3]. Further applications are contained in a forthcoming paper of the authors. We note in passing that the related strong unique continuation problem for second order elliptic operators is related to the eigenfunction bounds for the spherical harmonics. This was first observed in work of Jerison [6]; see also the authors paper [10] and further references therein.

In the next section we begin by considering the problem of obtaining dispersive and Strichartz estimates for the corresponding Schrödinger equation. This leads to an alternative proof of the eigenfunction bounds of Karadzhov [7] and Thangavelu [13]. Our approach has the advantage that is robust enough so that it allows us to obtain the same results with x^2 replaced by potentials in a very large class.

Then we direct our attention to the eigenfunctions $(-\Delta + x^2)\phi = \lambda^2\phi$. These are concentrated inside the ball $\{|x| \leq \lambda\}$, and have an exponential Airy type decay beyond this threshold. The behavior of the eigenfunctions inside the ball is not very different from (a rescaling of) what happens in a bounded domain. However, considerable care is required near the boundary of the ball, where the concentration scales are different. Consequently, it is more natural to obtain weighted L^p estimates with weights which are essentially powers of $\lambda - |x|$. The results we obtain strengthen those of Karadzhov and Thangavelu and complete the picture. As before, our methods are robust and apply equally to any potential which behaves roughly like x^2.

In the last section of the paper we construct appropriate examples which illustrate the possible concentration scales for eigenvalues of the Hermite operator and show that our L^p bounds are sharp.

To conclude the introduction we provide the reader with a special case of our main result. Denote the spectral projection to the eigenvalue $k = \lambda^2$ by P_k. Then our L^p bounds for eigenfunctions of the Hermite operator H have the form

$$(1) \qquad \|P_k\phi\|_{L^p} \lesssim k^{\rho(p)/2}\|\phi\|_{L^2}$$

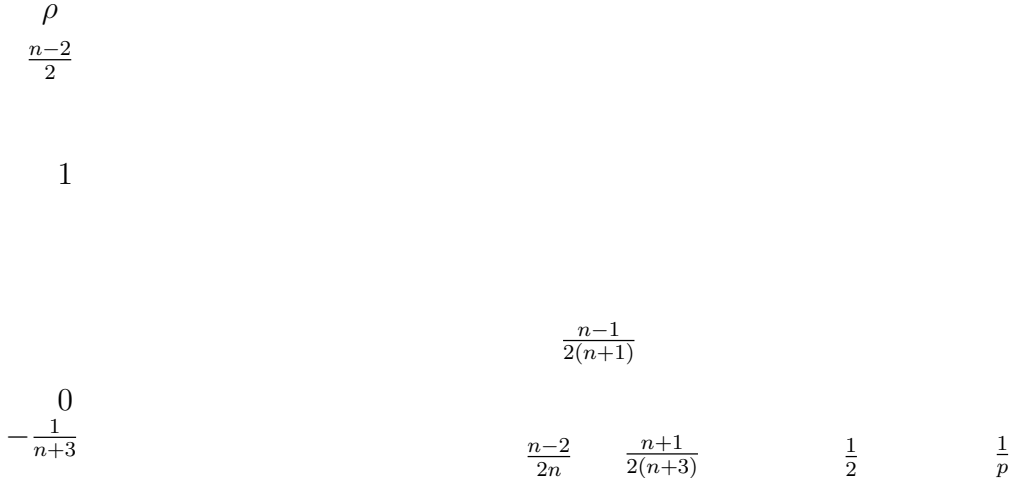where ρ is given as in the following figure.

FIGURE 1. The exponent ρ as function of $1/p$

2. Dispersive estimates for the Schrödinger equation

Consider the Schrödinger operator associated to the Hermite operator in $\mathbb{R}^n$,

$$i\partial_t - H$$

which generates a group of isometries $t \to e^{itH}$. Furthermore, this group is periodic with period π. Here we investigate the question of obtaining pointwise bounds for the kernel of e^{itH}. We also consider the same question for more general operators

$$H_V = -\Delta + V$$

Theorem 1. *a) The operators e^{itH} satisfy*

$$\|e^{itH}\|_{L^1 \to L^\infty} \lesssim |\sin t|^{-\frac{n}{2}}$$

b) Let V be a potential which satisfies

(2) $$|\partial^\alpha V| \leq c_\alpha, \qquad |\alpha| \geq 2$$

Then for $|t| \ll 1$ the operators e^{itH} satisfy

$$\|e^{itH_V}\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}}$$

Proof of Theorem 1. Prove the bound at time t_0. In case (a) the estimate follows from well known formulas for the Schrödinger kernel of the Hermite operator. We prefer to give a more flexible argument. By periodicity we can assume that $|t_0| \leq \pi/2$ and replace $|\sin t|$ by $|t|$. We rescale to time 1 by setting

$$t \to \frac{t}{t_0}, \qquad x \to \frac{x}{\sqrt{t_0}}$$

Then we need to prove an uniform bound for the kernel of $e^{iH^{t_0}}$, respectively $e^{iH_V^{t_0}}$ where

$$H^{t_0} = t_0^2 x^2 - \Delta, \qquad H_V^{t_0} = t_0 V(x\sqrt{t_0}) - \Delta$$

Since t_0 is in a bounded set, it follows that the symbols

$$h^{t_0}(x, \xi) = t_0^2 x^2 + \xi^2, \qquad h_V^{t_0}(x, \xi) = t_0 V(x\sqrt{t_0}) + \xi^2$$

satisfy the bounds

(3) $$|\partial_x^\alpha \partial_\xi^\beta h(x, \xi)| \leq c_{\alpha,\beta}, \qquad |\alpha| + |\beta| \geq 2$$

uniformly in $|t_0| \lesssim 1$. This implies that we can use directly Proposition 4.3 in [9] to obtain a phase space representation of the fundamental solution $K(t, y, \tilde{y})$ for $i\partial_t - H_V^{t_0}$. For the reader's convenience we restate the result here:

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Lemma 2.1 ([9]). *Let h be a symbol which satisfies (3). Then for $|t| \lesssim 1$ the fundamental solution $K(t, y, \tilde{y})$ for $i\partial_t - h^w(x, D)$ can be represented as*

$$(4) \qquad K(t, y, \tilde{y}) = \int_{\mathbb{R}^{2n}} e^{-\frac{1}{2}(\tilde{y}-x)^2} e^{-i\xi(\tilde{y}-x)} e^{i\psi(t,x,\xi)} e^{i\xi^t(y-x^t)} G(t, x, \xi, y) dx \, d\xi$$

where (x, ξ) to (x^t, ξ^t) is the Hamilton flow for h, the function G satisfies

$$(5) \qquad\qquad |(x^t - y)^\gamma \partial_x^\alpha \partial_\xi^\beta \partial_y^\nu G(t, x, \xi, y)| \lesssim c_{\gamma,\alpha,\beta,\nu}$$

and the real phase function ψ is determined by

$$\frac{d}{dt}\psi(t, x, \xi) = -h(x^t, \xi^t) + \xi^t h_\xi(x^t, \xi^t), \qquad \psi(0, x, \xi) = 0$$

A feature of the construction in [9] is that the integrand solves the evolution equation for each (x, ξ). In addition, it is concentrated in the phase space on the unit scale along the bicharacteristic $t \to (x^t, \xi^t)$. Such highly localized solutions are called wave packets. With this terminology, one can view the above lemma as a way of representing solutions for $i\partial_t - h^w(x, D)$ as almost orthogonal superpositions of wave packets.

In our case we need a pointwise bound for $K(1, y, \tilde{y})$. Neglecting all oscillations in (4) we write

$$|K(1, y, \tilde{y})| \lesssim \int_{\mathbb{R}^{2n}} e^{-\frac{1}{2}(\tilde{y}-x)^2} (1 + |y - x^1|)^{-N} dx \, d\xi$$

Due to (3) the Hamilton flow for $H_V^{t_0}$ is Lipschitz. Hence the integration in x is trivial, and we obtain

$$|K(1, y, \tilde{y})| \lesssim \int_{\mathbb{R}^n} (1 + |y - \tilde{y}^1|)^{-N} d\xi$$

In order to obtain an uniform bound it suffices to prove that the Lipschitz map $\xi \to \tilde{y}^1$ has a Lipschitz inverse. In the case of the Hermite operator this map is linear, and the desired conclusion is obtained by direct computation. For a more general potential V one computes the linearization of the Hamilton flow, which shows that

$$\frac{d\tilde{y}^1}{d\xi} = 2I_n + O(t_0)$$

For small t_0 this shows that the map $\xi \to \tilde{y}^1$ is a global diffeomorphism of $\mathbb{R}^n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

As a consequence of the dispersive estimates for the Schrödinger equation we also obtain Strichartz estimates for the Schrödinger equation:

Theorem 2. *Let V be a potential which satisfies*

$$(6) \qquad\qquad\qquad |\partial^\alpha V| \lesssim 1, \qquad |\alpha| = 2$$

Then the solution u to

$$(i\partial_t - H_V)u = f, \qquad u(0) = u_0$$

satisfies

$$\|u\|_{L^{p_1}(0,1;L^{q_1})} \lesssim \|u_0\|_{L^2} + \|f\|_{L^{p_2'}(0,1;L^{q_2'})}$$

whenever the pairs (p_1, q_1) and (p_2, q_2) are subject to

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \qquad 2 \le p \le \infty, \quad (n, p, q) \ne (2, 2, \infty)$$

Proof. a) If V satisfies the stronger bound (2) then this follows from Theorem 1 by standard arguments as in Ginibre-Velo [5] and the references therein. For the endpoint one can use the results in Keel-Tao [8].

b) For potentials V which only satisfy (6) we use a frequency decomposition of V. Given a smooth compactly supported function χ which equals 1 in a neighborhood of the origin we set

$$V = V_0 + V_1, \qquad V_0 = \chi(D)V, \quad V_1 = (1 - \chi(D)V)$$

The low frequency part V_0 satisfies (2), therefore we use part (a) of the proof. On the other hand the high frequency part V_1 is bounded, so we can add it in using the fact that the result in the theorem is stable with respect to L^2 bounded perturbations of H_V. $\qquad\square$

Using the Strichartz estimates one can easily obtain eigenfunction bounds. We begin with the Hermite operator H. Its spectrum is $n + 2\mathbb{N}$. A basis of orthogonal eigenfunction is given by the Hermite functions,

$$h_\alpha = (-1)^{|\alpha|} e^{x^2/2} \partial^\alpha e^{-x^2}$$

with

$$H h_\alpha = (n + 2|\alpha|) h_\alpha$$

For an integer $k = \lambda^2$ in the spectrum we denote by P_k the corresponding projector. For $\phi \in L^2$ the function

$$u = e^{itk} P_k \phi$$

solves $(i\partial_t - H)u = 0$. Hence we can apply Theorem 2 to obtain:

Corollary 2.2. *Let P_k be a spectral projector for H. Then*

$$\|P_k\phi\|_{L^p} \lesssim \|\phi\|_{L^2}, \qquad 2 \le p \le \frac{2n}{n-2}, \qquad (n, p) \ne (2, \infty)$$

In particular this gives L^p bounds for each eigenfunction. The estimate still holds in the case $(n, p) = (2, \infty)$, but to prove this one needs the stronger results in the next section.

In the case of the operator H_V the spectrum need not be discrete. Even if it is, there is no guarantee that there is an $O(1)$ spectral gap. Hence it is more useful to consider spectral projectors $P_{[k,k+1]}$ associated to unit size intervals. For $\phi \in L^2$ the function

$$u = e^{itk} P_{[k,k+1]} \phi$$

satisfies
$$\|(i\partial_t - H)u\|_{L^\infty L^2} \le \|\phi\|_{L^2}$$
Hence we can apply Theorem 2 to obtain:

Corollary 2.3. *Let V be a potential which satisfies* (6). *If $P_{[k,k+1]}$ is a spectral projector for H_V then*
$$\|P_{[k,k+1]}\phi\|_{L^p} \lesssim \|\phi\|_{L^2}, \qquad 2 \le p \le \frac{2n}{n-2}, \qquad (n,p) \ne (2,\infty)$$

Here and in the sequel we formulate results for $n \ge 2$, which are valid for $n = 1$ with modifications which are either obvious or discussed.

3. WEIGHTED L^p EIGENFUNCTION BOUNDS

Let (ϕ, λ^2) be an eigenfunction, respectively an eigenvalue for the Hermite operator. The function ϕ is essentially concentrated in the ball $\{|x| \le \lambda\}$ (modulo an exponentially decaying tail). We split the interior of this region into overlapping dyadic parts with respect to the distance to the boundary,
$$D_j^{int} = \{|x| \in [\lambda(1 - 2^{-2(j-1)}), \lambda(1 - 2^{-2(j+1)})]\}, \qquad 1 \le 2^j \le \lambda^{\frac{2}{3}}$$
By D^{bd} we denote a narrow strip near the boundary of the ball,
$$D^{bd} = \{||x| - \lambda| \le \lambda^{-\frac{1}{3}}\}$$
Finally, D^{ext} is the exterior of the ball,
$$D^{ext} = \{|x| > \lambda + \frac{1}{2}\lambda^{-\frac{1}{3}}\}$$

The symbol of $x^2 - \Delta - \lambda^2$ equals
$$x^2 + \xi^2 - \lambda^2$$
If $x \in D_j^{int}$ then this symbol can be zero only in the region
$$|\xi| \approx \lambda 2^{-j}$$
Hence the case $2^j \approx \lambda^{\frac{2}{3}}$ corresponds to an annulus of thickness $\lambda^{-\frac{1}{3}}$ and to $|\xi| \approx \lambda^{\frac{1}{3}}$ which is exactly on the scale of the uncertainty principle. This explains the limitation for the range of 2^j. It also gives the scale on which ϕ decays away from the ball.

Given an eigenvalue λ we define the spaces $l_\lambda^q L^p$ of functions in $\mathbb{R}^n$ with norm
$$\|f\|_{l_\lambda^q L^p}^q = \|f\|_{L^p(D^{ext})}^q + \|f\|_{L^p(D^{bd})}^q + \sum_{1 \le j, 2^j \le \lambda^{2/3}} \|f\|_{L^p(D_j^{int})}^q$$
with the usual modification when $q = \infty$. The subscript λ is used because the sets D_j^{int}, D^{bd} and D^{ext} depend on λ.

For $x \in \mathbb{R}^n$ we denote
$$y = \lambda^{-\frac{2}{3}}(\lambda^2 - x^2), \qquad \langle y \rangle_- = 1 + y_-, \qquad \langle y \rangle_+ = 1 + y_+.$$

Then

Theorem 3. *a) Let* $2 \leq p \leq \frac{2(n+1)}{n-1}$. *Then*

(7) $\quad \|\lambda^{\frac{1}{3}-\frac{n}{3}(\frac{1}{2}-\frac{1}{p})}\langle y\rangle_+^{-\frac{1}{4}+\frac{n+3}{4}(\frac{1}{2}-\frac{1}{p})}\langle y\rangle_-^{1-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}\phi\|_{l_\lambda^\infty L^p} \lesssim \|\phi\|_{L^2} + \|(H-\lambda^2)\phi\|_{L^2}$

b) Let $\frac{2(n+1)}{n-1} \leq p \leq \infty$. *Then*

(8) $\qquad\qquad \|\lambda^{\frac{1}{3}-\frac{n}{3}(\frac{1}{2}-\frac{1}{p})}\langle y\rangle_+^{\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}\langle y\rangle_-^N P_{\lambda^2}\phi\|_{l_\lambda^\infty L^p} \lesssim \|\phi\|_{L^2}$

In the interesting regions D_j the estimates can be reformulated as follows. If $2 \leq p \leq \frac{2(n+1)}{n-1}$ then

(9) $\qquad \lambda^{\frac{1}{2}-\frac{1}{p}}2^{\frac{j}{2}(1-(n+3)(\frac{1}{2}-\frac{1}{p}))}\|\phi\|_{L^p(D_j^{int})} \lesssim \|\phi\|_{L^2} + \|(H-\lambda^2)\phi\|_{L^2}$

and if $\frac{2(n+1)}{n-1} \leq p \leq \infty$ then

(10) $\qquad\qquad \lambda^{1-n(\frac{1}{2}-\frac{1}{p})}2^{j(1-n(\frac{1}{2}-\frac{1}{p}))}\|P_{\lambda^2}\phi\|_{L^p(D_j^{int})} \lesssim \|\phi\|_{L^2}$

The estimates in Theorem 3 are sharp. We give here a heuristic motivation, and complement this with a more precise (yet not as general) construction in the last section of this paper.

The symbol of H is $x^2 + \xi^2$. The λ^2 eigenfunctions of H are concentrated in the phase space within a neighbourhood of size 1 of the characteristic set $\{x^2 + \xi^2 = \lambda^2\}$. For a more precise description, one can consider the Hamilton flow for H restricted to the same sphere. It is periodic, and one can can construct eigenfunctions which are concentrated in the phase space within a neighbourhood of size 1 of each such bicharacteristic. To obtain eigenfunctions which are pointwise larger one may consider superpositions of such eigenfunctions corresponding to neighbouring bicharacteristics. However, such a concentration can only occur on a smaller set in the physical space; the optimal balance between the amplitude and the localization region is dictated by the uncertainty principle.

Consider first the estimate (7). Within D_1^{int} the worst concentration occurs for eigenvalues concentrated both in space and in frequency in a 1 neighborhood of a single bicharacteristic. For an arbitrary j one needs to consider modes which within D_j^{int} are spatially concentrated within $2^{-\frac{j}{2}}$ of a bicharacteristic, with a dual concentration in frequency. In the extreme case, for $2^j = \lambda^{\frac{2}{3}}$, the concentration occurs as a bump in a $\lambda^{-\frac{1}{3}}$ ball near the circle $|x| = \lambda$. Except in dimension 1, this corresponds to a solution which is uniformly spread in an ellipsoid of size $\lambda \times \lambda^{\frac{1}{3}}$.

A different behavior is responsible for estimate (8). Here the concentration corresponds to an equidistribution of energy between all bicharacteristics through a point P. For $P \in D_j^{int}$ this leads to concentration in a $2^j\lambda^{-1}$ ball

around the point. If for instance P is the origin then this could be a spherically symmetric mode.

Remark 3.1. *We expect that the estimate holds with $l_\lambda^\infty L^p$ replaced by L^p for all $p > 2$. This is because the concentration cannot occur simultaneously in many regions D_j^{int}. From Corollary 2.2 we already know that the improvement holds if $\frac{2n}{n-2} \leq p \leq \infty$. On the other hand, such an improvement cannot be true for $p = 2$.*

If we want to write an estimate without the weights then the sign of the power of $\langle y \rangle_+$ becomes important:

Corollary 3.2. *Let (ϕ, λ^2) be an eigenfunction, respectively an eigenvalue for the Hermite operator. Then*

$$\|\phi\|_{L^p} \lesssim \lambda^{\rho(p)} \|\phi\|_{L^2}$$

where for $n \geq 2$ we have (see Figure 1)

$$\rho(p) = \begin{cases} -(\frac{1}{2} - \frac{1}{p}) & 2 \leq p < \dfrac{2(n+3)}{n+1} \\[2ex] -\dfrac{1}{3} + \dfrac{n}{3}(\dfrac{1}{2} - \dfrac{1}{p}) & \dfrac{2(n+3)}{n+1} < p \leq \dfrac{2n}{n-2} \\[2ex] -1 + n(\dfrac{1}{2} - \dfrac{1}{p}) & \dfrac{2n}{n-2} \leq p \leq \infty \end{cases}$$

while for $n = 1$

$$\rho(p) = \begin{cases} -(\dfrac{1}{2} - \dfrac{1}{p}) & 2 \leq p < 4 \\[2ex] -\dfrac{1}{3} + \dfrac{1}{3}(\dfrac{1}{2} - \dfrac{1}{p}) & 4 < p \leq \infty \end{cases}$$

In the first and the third case the worst bound is in D_1^{int}, while in the second case the worst bound is within D^{bd}.

Endpoints discussion: The arguments below yield the $p = \infty$ endpoint even in dimension $n = 1, 2$. For $n \geq 3$ the case $p = \frac{2n}{n-2}$ comes from the Strichartz estimates in the previous section. The $p = \frac{2(n+3)}{n+1}$ endpoint is false for $n = 1$, where the eigenvalues are simple and have an Airy type behavior at the ends. However, it is likely be true for $n \geq 2$, because it is not possible to have concentration at all scales.

Proof of Theorem 3: The L^2 bound. This is the key step in the proof of Theorem 3, because the L^2 bound is strong enough to provide the localization which is needed for the rest of the arguments. We assume that

$$\|\phi\|_{L^2} + \|(H - \lambda^2)\phi\|_{L^2} = 1$$

and we shall prove the following bounds:

$$\|\phi\|_{L^2(D_j^{int})} \lesssim 2^{-\frac{j}{2}}, \qquad \|\nabla\phi\|_{L^2(D_j^{int})} \lesssim \lambda 2^{-\frac{3j}{2}}$$

(11) $$\|\phi\|_{L^2(D^{bd})} \lesssim \lambda^{-\frac{1}{3}}, \qquad \|\nabla\phi\|_{L^2(D^{bd})} \lesssim 1$$

$$\|(x^2-\lambda^2)\phi\|_{L^2(D^{ext})} \lesssim \lambda^{\frac{1}{3}}, \qquad \|(x^2-\lambda^2)^{\frac{1}{2}}\nabla\phi\|_{L^2(D^{ext})} \lesssim \lambda^{\frac{1}{3}}$$

We first offer some intuitive justification for these bounds. The one in D^{ext} is an elliptic estimate. For the rest, it is convenient to think of an almost orthogonal basis for the eigenspaces of H which consists of eigenfunctions which are localized in the phase space within a neighbourhood of size 1 of null bicharacteristics of $H - \lambda^2$. The energy of such eigenvalues is equidistributed along the corresponding bicharacteristics, which have length λ. Hence in order to measure what portion of the energy is contained in D_j^{int} it suffices to measure the length of the bicharacteristic within D_j^{int}, which is at most $2^{-j}\lambda$. This justifies the L^2 bounds. For the gradient bounds we simply note that on the characteristic set of $H - \lambda^2$ within D_j we have $|\xi| \approx 2^{-j}\lambda$.

To prove (11) we begin with some simpler estimates. First note that

$$\langle \phi, (H-\lambda^2)\phi \rangle = \|\nabla\phi\|_{L^2}^2 + \langle (x^2-\lambda^2)\phi, \phi \rangle$$

Since $x^2 - \lambda^2 \geq -\lambda^2$, this easily leads to

(12) $$\|\nabla\phi\|_{L^2} \lesssim \lambda$$

This argument can be easily improved. Choose

$$W = ((\lambda^2-x^2)^2 + \lambda^{\frac{4}{3}})^{-\frac{1}{2}}$$

Then an integration by parts yields

$$\langle W\phi, (H-\lambda^2)\phi \rangle = \|W^{\frac{1}{2}}\nabla\phi\|_{L^2}^2 + O(\|\phi\|_{L^2})$$

which produces another nonsharp estimate,

(13) $$\|((\lambda^2-x^2)^2 + \lambda^{\frac{4}{3}})^{-\frac{1}{4}}\nabla\phi\|_{L^2} \lesssim 1$$

which is slightly better than (12) near the sphere $|x| = \lambda$. We use this weaker bound to eliminate some error terms in the estimates which follow.

We will argue in a manner which is similar to the Carleman estimates. First we introduce a bounded exponential weight which does not change the estimates, but allows us to replace the operator $H - \lambda^2$ with its conjugate with respect to the weight, namely the operator $H_{a,\lambda}$ below. The advantage is that the conjugate operator is no longer selfadjoint; precisely, the gain in the L^2 estimates comes from the positivity of the commutator between its self-adjoint and its skew-adjoint part. In order to guarantee this the weight needs to be chosen roughly so that it is convex along the null bicharacteristics for $H - \lambda^2$ near the ball $\{|x| = \lambda\}$.

An alternate approach would be to obtain a Morawetz type estimate using a suitably chosen multiplier. We do not pursue this as it seems slightly less precise, and requires more care in the error estimates.

Consider a weight function $a : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) a is nondecreasing and equals 0 in $(-\infty, -2]$.

(ii) a and its derivatives satisfy the bounds

(14) $$|\partial^k a(y)| \lesssim (1 + |y|)^{\frac{1}{2}-k}, \qquad k \geq 0$$

Let c be a small positive constant which will be chosen later. All implicit constants in the estimates that follow will be independent of c. Denote

$$\psi(x) = e^{-\lambda^{-\frac{2}{3}} a(y)} \phi(x), \qquad y = c\lambda^{-\frac{2}{3}}(\lambda^2 - x^2)$$

Introduce also the conjugated operator

$$H_{a,\lambda} = e^{-\lambda^{-\frac{2}{3}} a(y)}(H - \lambda^2)e^{\lambda^{-\frac{2}{3}} a(y)}$$

Then

$$H_{a,\lambda}\psi = e^{-\lambda^{-\frac{2}{3}} a(y)}(H - \lambda^2)\phi$$

The exponential weight is bounded since $x^2 \geq 0$ implies $y \leq \lambda^{\frac{4}{3}}$. Hence

$$\|\phi\|_{L^2} + \|(H - \lambda^2)\phi\|_{L^2} \approx \|\psi\|_{L^2} + \|H_{a,\lambda}\psi\|_{L^2}$$

On the other hand, given the above properties of a it is easy to verify that ψ satisfies (11) if ϕ does. Hence we have replaced ϕ and $H - \lambda^2$ by ψ, respectively $H_{a,\lambda}$.

The conjugated operator $H_{a,\lambda}$ is decomposed into a selfadjoint and a skew-adjoint part,

$$H_{a,\lambda} = H_{a,\lambda}^{re} + H_{a,\lambda}^{im}$$

$$H_{a,\lambda}^{re} = -\Delta + x^2 - \lambda^2 - 4c^2\lambda^{-\frac{8}{3}}x^2|a'(y)|^2$$

$$H_{a,\lambda}^{im} = -2c\lambda^{-\frac{4}{3}}(\partial x a'(y) + a'(y)x\partial)$$

where we use a short notation for operators $\partial x a'\phi = \sum_{i=1}^n \partial_{x_i}(x_i a'\phi)$.

Ideally we would like the commutator $\{H_{a,\lambda}^{re}, H_{a,\lambda}^{im}\}$ to be positive definite. However, this is too much to hope for. Instead, it turns out that we can choose the exponential weight so that the commutator has a positive symbol only near the characteristic set of $H_{a,\lambda}$. To compensate for that we introduce a real correction term W_λ of the form

$$W_\lambda = 4c^2 b(c\lambda^{-\frac{2}{3}}(\lambda^2 - x^2))$$

where the positive function b is chosen to satisfy the bounds

(15) $$|\partial^k b(y)| \lesssim \langle y \rangle_- \langle y \rangle_+^{-\frac{3}{2}} \langle y \rangle^{-k}, \qquad k \geq 0$$

Then we compute

$$\begin{aligned}
\|H_{a,\lambda}\psi\|_{L^2}^2 &= \|H_{a,\lambda}^{re}\psi\|_{L^2}^2 + \|H_{a,\lambda}^{im}\psi\|_{L^2}^2 + \langle [H_{a,\lambda}^{re}, H_{a,\lambda}^{im}]\psi, \psi \rangle \\
&= \|(H_{a,\lambda}^{re} - W_\lambda)\psi\|_{L^2}^2 + \|H_{a,\lambda}^{im}\psi\|_{L^2}^2 + \langle C\psi, \psi \rangle
\end{aligned}$$

where

$$C = [H_{a,\lambda}^{re}, H_{a,\lambda}^{im}] + H_{a,\lambda}^{re}W_\lambda + W_\lambda H_{a,\lambda}^{re} - W_\lambda^2$$

Therefore we obtain

(16) $$\langle C\psi, \psi \rangle \leq \|H_{a,\lambda}\psi\|_{L^2}^2 \lesssim 1$$

Our goal is to choose the weights a and b so that C is a positive operator which controls the norms in (11) for ψ. Hence we need to compute C. This can be rather tedious, and in order to simplify the analysis we make two conventions:

(a) We discard all derivative terms which can be controlled by (13).

(b) All the scalar terms which are negligible are incorporated into a generic term called "error", which satisfies

$$|\text{error}| \lesssim 1 + c^2\lambda^{\frac{2}{3}}\langle y \rangle_+^{-1}\langle y \rangle_-^2$$

As we shall see later on, the error term in (b) is easily controlled by the main term in (17) below. For instance the scalar W_λ^2 is an error term. We consider the two remaining terms in C. Using the bounds on the derivatives of a in (ii) above yields

$$[H_{a,\lambda}^{re}, H_{a,\lambda}^{im}] = 8c(\lambda^{-\frac{4}{3}}x^2 a'(y) - 2c\lambda^{-2}\partial x a''(y)x\partial + \lambda^{-\frac{4}{3}}\partial a'(y)\partial) + \text{error}$$

On the other hand

$$W_\lambda H_{a,\lambda}^{re} + H_{a,\lambda}^{re}W_\lambda = -8c^2(\partial b(y)\partial + (x^2 - \lambda^2)b(y)) + \text{error}$$

Summing up the two sets of estimates and using Cauchy-Schwartz we obtain

$$1 \gtrsim \int \left(8c^2 b(y) - 16c^2\lambda^{-2}x^2 a''_-(y) - 8c\lambda^{-\frac{4}{3}}a'(y)\right)|\nabla\phi|^2$$
$$+ \left(8c\lambda^{-\frac{4}{3}}x^2 a'(y) - 8c^2 b(y)(x^2 - \lambda^2) + \text{error}\right)|\phi|^2 dx$$

where a''_- is the negative part of a'', $a''_- = (|a''| - a'')/2$.

Using the estimates (12) and (13) we can control the third derivative term, and also replace the x^2 in the second derivative term by λ^2. The x^2 in the first scalar term can be replaced by λ^2 modulo a bounded remainder. Thus we obtain

(17) $$1 \gtrsim \int \left(c^2(b(y) - 2a''_-(y))\right)|\nabla\phi|^2 + \left(c\lambda^{\frac{2}{3}}(a'(y) - yb(y)) + \text{error}\right)|\phi|^2 dx$$

To conclude the proof we need to choose the functions a and b so that the two coefficients above are nonnegative and sufficiently large. The key step is summarized in the next Lemma.

Lemma 3.3. *Let $(\varepsilon_k)_{k\geq 1}$ be a slowly varying positive sequence with the property that $\sum \varepsilon_k = 1$. Then there are functions a, b as above so that in addition we have:*

$$b(y) - 2a''_-(y) \gtrsim \begin{cases} (1 + |y|) & y < 1 \\ \\ \varepsilon_k|y|^{-\frac{3}{2}} & y \in [2^k, 2^{k+1}] \end{cases}$$

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$$a'(y) - yb(y) \gtrsim \begin{cases} (1 + y^2) & y < 1 \\ \\ \varepsilon_k |y|^{-\frac{1}{2}} & y \in [2^k, 2^{k+1}] \end{cases}$$

We first show how to conclude the proof of the L^2 estimates (11). First we need to control the error term, and this is where we use the freedom to choose c sufficiently small. This is because the positive scalar term in (17) has a factor of c, while the unbounded part of the error has a factor of c^2. Hence in order to control the error it suffices to insure that

$$\varepsilon_k \gtrsim 2^{-\frac{k}{2}}$$

which guarantees that

$$a'(y) - yb(y) \gtrsim |y|^{-1}$$

The bounds in the lemma suffice to obtain (11) in the regions D^{ext} and D^{bd}. In order to prove (11) in the region D_j^{int} we need to also choose the ε_k's so that

$$\varepsilon_k \approx 1, \qquad \text{when } 2^{k+2j} \approx \lambda^{\frac{4}{3}}$$

Proof of Lemma 3.3. (i) The range $y < 0$. Observe first that we can add any positive function to b for $y < 0$ and improve both inequalities. Hence as long as $a'(0) > 0$ and $a''(0) \geq 0$ we simply need to choose b sufficiently large and growing like $|y|$ at $-\infty$.

(ii) The range $y \geq 0$. Here it suffices to choose a so that $a' > 0$, $a''(y) < 0$ for large y, $a''(0) \geq 0$ and

$$a'(y) - 2ya''(y) \gtrsim \varepsilon_k |y|^{-\frac{1}{2}} \qquad y \in [2^k, 2^{k+1}]$$

Then b can be chosen $b(y) = a''(y) + (2y)^{-1}a'(y)$ for large y and arbitrary between $2a''_-(y)$ and $y^{-1}a'(y)$ for small y.

It remains to describe the choice of a'. We begin with a simple choice, namely

$$a'_0(y) = (1 + y^2)^{-\frac{1}{4}}$$

which satisfies all conditions except for a weaker bound

$$a'_0(y) - 2ya''_0(y) \gtrsim (1 + y^2)^{-\frac{5}{4}}$$

Then we choose a nonincreasing function d with $\frac{1}{2} \leq d \leq \frac{3}{2}$ so that $d'(0) = 0$ and

$$d'(y) \sim -\varepsilon_k y^{-1} \qquad y \in [2^k, 2^{k+1}]$$

Finally, we set

$$a'(y) = d(y)a'_0(y)$$

Then in $[2^k, 2^{k+1}]$ we have

$$a'(y) - 2ya''(y) = d(y)(a'_0(y) - 2ya''_0(y)) - 2ya'_0(y)d'(y) \gtrsim \varepsilon_k |y|^{-\frac{1}{2}}$$

$\square$

Proof of Theorem 3: the $L^{\frac{2(n+1)}{n-1}}$ bound. We split D_j^{int} into balls $D_j^{int,k}$ of radius $r_j = \lambda 2^{-2j}$ and consider a corresponding partition of unity

$$1 = \chi^{bd} + \chi^{ext} + \sum_{j,k} \chi_j^k(x)$$

with

$$|\nabla \chi_j^k(x)| \lesssim r_j^{-1}, \qquad |\nabla^2 \chi_j^k(x)| \lesssim r_j^{-2}$$

The localized pieces of ϕ are

$$\phi_j^k = \chi_j^k \phi, \qquad \phi^{bd} = \chi^{bd} \phi, \qquad \phi^{ext} = \chi^{ext} \phi$$

We consider three cases.

   a) The interior estimate. We claim that the functions ϕ_j^k satisfy

(18)
$$\sum_k 2^j \|\phi_j^k\|_{L^2}^2 + 2^{-j} \|(H - \lambda^2)\phi_j^k\|_{L^2}^2 \lesssim 1$$

The first part follows directly from the lemma, for the second we commute

$$(H - \lambda^2)\phi_j^k = [-\Delta, \chi_j^k]\phi = \nabla \chi_j^k \nabla \phi + \nabla^2 \chi_j^k \phi$$

and use the lemma and the bounds on χ_j^k.

   To obtain L^p bounds for each of these pieces we use a Strichartz type estimate which is a special case of Theorem 2, [9]. For convenience we state it in the following

Lemma 3.4. *Let W be a real potential in the unit ball which satisfies $W \sim 1$ and*

$$|\partial^\alpha W| \lesssim 1, \qquad |\alpha| = 1, 2$$

and let a^{ij} be elliptic coefficients of class C^2. Then for all $\mu > 1$ and u supported in the unit ball we have

$$\|u\|_{L^{\frac{2(n+1)}{n-1}}} \lesssim \mu^{-\frac{1}{n+1}}(\mu^{\frac{1}{2}}\|u\|_{L^2} + \mu^{-\frac{1}{2}}\|(a^{ij}\partial_i\partial_j + \mu^2 W)u\|_{L^2})$$

Discussion. Strictly speaking, in order to apply Theorem 2, [9] we need the additional bounds

$$|\partial^\alpha W| \lesssim \mu^{\frac{|\alpha|-2}{2}}, \qquad |\alpha| \geq 2$$

But such bounds can be easily gained by truncating the potential W in frequency,

$$W = W_0 + W_1 \qquad W_0 = \chi(\mu^{-\frac{1}{2}}D)W, \quad W_1 = (1 - \chi(\mu^{-\frac{1}{2}}D))W$$

Then the low frequency part W_0 satisfies the stronger bounds. The high frequency part on the other hand satisfies a pointwise bound $|W_1| \lesssim \mu^{-1}$ therefore it does not affect the size of the right hand side. Another observation is that Theorem 2, [9] only gives the $L^{\frac{2(n+1)}{n-1}}$ bound for u in the frequency region $|\xi| \lesssim \mu$. However, outside this region the symbol $-\xi^2 + \mu^2 W$ is elliptic, therefore even stronger bounds are easy to obtain. $\qquad\square$

Given the L^2 bound (18), the $L^{\frac{2(n+1)}{n-1}}$ estimate for ϕ in D_j^{int} would follow from an estimate for ϕ_j^k, namely

$$\|\phi_j^k\|_{L^{\frac{2(n+1)}{n-1}}} \lesssim (2^j \lambda^{-1})^{\frac{1}{n+1}} (2^j \|\phi_j^k\|_{L^2} + 2^{-j}\|(-\Delta + x^2 - \lambda^2)\phi_j^k\|_{L^2})$$

This follows from Lemma 3.4 after rescaling to the unit spatial scale. Note that within $D_j^{int,k}$ the symbol $x^2 + \xi^2 - \lambda^2$ is elliptic at frequencies $|\xi| \gg 2^{-j}\lambda$, so the interesting region in phase space has size $(2^{-2j}\lambda)^n \times (2^{-j}\lambda)^n$. After rescaling, the frequency becomes $\mu = 2^{-3j}\lambda^2 > 1$.

b) The boundary estimate. We can use the bounds in (11) to obtain

$$\lambda^{\frac{1}{3}}\|\phi^{bd}\|_{L^2} + \|\nabla\phi^{bd}\|_{L^2} \lesssim 1$$

Then the $L^{\frac{2(n+1)}{n-1}}$ bound for ϕ^{bd} is straightforward by Sobolev embeddings.

c) The exterior estimate. The bound in (11) implies that

$$\|(x^2 - \lambda^2)\phi^{ext}\|_{L^2} + \|(x^2 - \lambda^2)^{\frac{1}{2}}\nabla\phi^{ext}\|_{L^2} \lesssim \lambda^{\frac{1}{3}}$$

Then by (weighted) Sobolev embeddings we obtain

$$\|(x^2 - \lambda^2)^{\frac{n+2}{2(n+1)}}\nabla\phi^{ext}\|_{L^{\frac{2(n+1)}{n-1}}} \lesssim \lambda^{\frac{1}{3}}$$

$\square$

Proof of Theorem 3: the L^∞ bound. We consider separately two cases.

(i) In D^{int} and D^{bd} we only use the size of the potential. Each region $D_j^{int,k}$ has size $r_j = 2^{-2j}\lambda$ and corresponds to frequencies of size $\mu_j = 2^{-j}\lambda$. In $D_j^{int,k}$ we have an elliptic estimate,

$$\mu_j^{n(\frac{1}{q}-\frac{1}{p})}\|\chi_j^k\phi\|_{L^q} \lesssim \mu_j^{-2}\|\Delta\phi\|_{L^p(D_j^{int,k})} + \|\phi\|_{L^p(D_j^{int,k})}, \qquad 1 < p \le q \le \frac{2p}{n-2p}$$

which follows by rescaling from the case $r = 1$, $\mu \ge 1$.

We replace $\Delta\phi$ with $(\Delta - x^2 + \lambda^2)\phi + (x^2 - \lambda^2)\phi$, begin with $p = \frac{2(n+1)}{n-1}$ and apply the above estimate iteratively until we arrive at $q = \infty$. The same idea works also in D^{bd}.

(ii) In D^{ext} we also use the favorable sign of the potential. We first improve the L^2 bound on ϕ, namely the last part of (11). We rewrite (11) as

$$\|\langle y\rangle_+ \phi\|_{L^2(D^{ext})} \lesssim \lambda^{-\frac{1}{3}}$$

and inductively show that

(19)
$$\|\langle y\rangle_+^{\frac{N}{2}}\phi\|_{L^2(D^{ext})} \lesssim \lambda^{-\frac{1}{3}}$$

Suppose that (19) holds. Then we compute

$$\int \chi^{ext}\langle y\rangle_+^{\frac{N}{2}}\phi \langle y\rangle_+^{\frac{N}{2}}(H - \lambda^2)\phi\, dx = \int \chi^{ext}\langle y\rangle_+^N((x^2 - \lambda^2)\phi^2 + |\nabla\phi|^2)$$
$$+ \chi^{ext}\langle y\rangle_+^{N-1}\lambda^{-\frac{2}{3}}\phi x\nabla\phi + \nabla\chi^{ext}\langle y\rangle_+^N\phi\nabla\phi\, dx$$

In the region $y \gg 1$ the last two right hand side terms are controlled by the first. In the region $y \sim 1$ we use (11). Together with (19) this yields

$$\int \chi^{ext} \langle y \rangle_+^N ((x^2 - \lambda^2)\phi^2 + |\nabla \phi|^2) dx \lesssim 1$$

which implies (19) with N replaced by $N + 1$.

At this point we can conclude the proof as in case (i). Precisely, to fix the size of the potential we also consider a covering with balls

$$D^{ext} \subset \bigcup_{j \geq 1, k} D_j^{ext,k}$$

and a corresponding partition of unit

$$1 = \sum \chi_j^k \qquad \text{in } D^{ext}$$

so that in $D_j^{ext,k}$ we have

$$|x|^2 - \lambda^2 \sim 2^{2j} \lambda^{\frac{2}{3}}$$

In each of these balls we use the same elliptic argument as in (i). $\qquad \square$

4. Extensions

In the previous section we used the potential x^2. However, its precise form does not play a fundamental role in the estimates. Here we consider instead the operator H_V with a positive potential V which satisfies the following conditions:

$$(20) \qquad\qquad V \sim |x|^2, \qquad |\nabla V| \sim |x|, \qquad |\partial_x^2 V| \lesssim 1$$

Given an eigenvalue λ^2 of V we introduce as before the dyadic regions

$$D_j^{int} = \{\lambda^2 - V \in [2^{-2(j-1)}\lambda^2, 2^{-2(j+1)}\lambda^2] \qquad 2 \leq 2^j \leq \lambda^{\frac{2}{3}}\}$$

$$D^{bd} = \{|\lambda^2 - V| \lesssim \lambda^{\frac{2}{3}}\} \qquad D^{ext} = \{V > \lambda^2 + \lambda^{\frac{2}{3}}\}$$

Also we set

$$y = \lambda^{-\frac{2}{3}}(V - \lambda^2)$$

Then

Theorem 4. *a) Let* $2 \leq p \leq \frac{2(n+1)}{n-1}$. *Then*

$$(21) \quad \|\lambda^{\frac{1}{3} - \frac{n}{3}(\frac{1}{2} - \frac{1}{p})} \langle y \rangle_-^{-\frac{1}{4} + \frac{n+3}{4}(\frac{1}{2} - \frac{1}{p})} \langle y \rangle_+^{1 - \frac{n}{2}(\frac{1}{2} - \frac{1}{p})} \phi\|_{l_\chi^\infty L^p} \lesssim \|\phi\|_{L^2} + \|(H_V - \lambda^2)\phi\|_{L^2}$$

b) Let $\frac{2(n+1)}{n-1} \leq p \leq \infty$. *Then*

$$(22) \qquad \|\lambda^{\frac{1}{3} - \frac{n}{3}(\frac{1}{2} - \frac{1}{p})} \langle y \rangle_-^{\frac{1}{2} - \frac{n}{2}(\frac{1}{2} - \frac{1}{p})} \langle y \rangle_+^N P_{[\lambda^2, \lambda^2 + 1]} \phi\|_{l_\chi^\infty L^p} \lesssim \|\phi\|_{L^2}$$

The result in Corollary 3.2 also applies to H_V.

Proof. (Sketch) We only need the counterpart of (11), the rest of the proof is identical.

Without any restriction in generality we can replace V with a mollified potential $\chi(D)V$, since the difference is bounded. This allows us to use the additional assumptions (2).

We use the same functions a and b as in the proof of (11) but with the above definition of y. Now the selfadjoint and skew-adjoint parts of the conjugated operator are

$$H_{a,\lambda}^{re} = -\Delta + V - \lambda^2 - c^2\lambda^{-\frac{6}{3}}|\nabla V|^2|a'(y)|^2$$

$$H_{a,\lambda}^{im} = -c\lambda^{-\frac{4}{3}}(\partial\nabla V a'(y) + a'(y)\nabla V\partial)$$

The correction term is chosen to be

$$W_\lambda = 2c^2\lambda^{-2}|\nabla V|^2 b(y)$$

Modulo error terms the operator C has the form

$$\begin{aligned}
C &= 2c\left(\lambda^{-\frac{4}{3}}|\nabla V|^2 a'(y) - \lambda^{-2}b(y)|\nabla V|^2(V - \lambda^2)\right) + 2c^2\partial(\lambda^{-2}|\nabla V|^2 b(y) \\
&+ \lambda^{-2}\nabla V a''(y)\nabla V)\partial
\end{aligned}$$

Then the argument continues as in the proof of (11).

$\square$

5. REMARKS ON OPTIMALITY

We begin by recalling bounds and expansions of Hermite functions. The Hermite functions are the eigenfunctions of the one dimensional Hermite operator, and solve

$$(23) \qquad\qquad -h_k'' + x^2 h_k = (2k+1)h_k$$

for nonnegative integers k. They are given by

$$h_k(x) = e^{x^2/2}(-1)^k\frac{d^k}{dx^k}e^{-x^2}.$$

and are even functions for even k and odd functions for odd k. Here the meaning of k differs slightly from the previous sections. In dimension n a complete set of eigenfunctions is given by

$$h_\alpha(x) = \prod_{i=1}^n h_{\alpha_i}(x_i)$$

where the corresponding eigenvalue is $n + 2|\alpha|$.

To construct highly localized eigenfunctions we need a better understanding of the behavior of the Hermite functions. This is well understood by now, and we describe it next.

The ODE (23) has a turning point at $x = \sqrt{2k+1}$. We set $\lambda = \sqrt{2k+1}$. Then the functions h_k have an oscillatory behavior for small x, an Airy type

behavior for $|x|$ close to λ and Gaussian decay for large x. More precisely, define

$$(24) \qquad s^-(x) = \int_0^x \sqrt{|t^2 - \lambda^2|}\, dt,$$

$$(25) \qquad s^+(x) = \int_\lambda^x \sqrt{|t^2 - \lambda^2|}\, dt,$$

Then

Lemma 5.1. *The normalized eigenfunctions*

$$\tilde{h}_k = h_k \|h_k\|_{L^2}^{-1}$$

satisfy

$$\tilde{h}_{2k} = \begin{cases} a_{2k}^-(\lambda^2 - x^2)^{-\frac{1}{4}}(\cos s^-(x) + error) & |x| < \lambda - \lambda^{-\frac{1}{3}} \\ O(\lambda^{\frac{1}{6}}) & \lambda - \lambda^{-\frac{1}{3}} \le x \le \lambda + \lambda^{-\frac{1}{3}} \\ a_{2k}^+ e^{-s^+(x)}(\lambda^2 - x^2)^{-\frac{1}{4}}(1 + error) & |x| > \lambda + \lambda^{-\frac{1}{3}} \end{cases}$$

$$\tilde{h}_{2k+1} = \begin{cases} a_{2k+1}^-(\lambda^2 - x^2)^{-\frac{1}{4}}(\sin s^-(x) + error) & |x| < \lambda - \lambda^{-\frac{1}{3}} \\ O(\lambda^{\frac{1}{6}}) & \lambda - \lambda^{-\frac{1}{3}} \le x \le \lambda + \lambda^{-\frac{1}{3}} \\ a_{2k+1}^+ e^{-s^+(x)}(\lambda^2 - x^2)^{-\frac{1}{4}}(1 + error) & x > \lambda + \lambda^{-\frac{1}{3}} \end{cases}$$

where

$$|a_k^\pm| \sim 1, \qquad error = O((|x^2 - \lambda^2|^{-\frac{1}{2}}||x| - \lambda|^{-1})$$

Note that the error term is $O(1)$ if $||x| - \lambda| \sim \lambda^{-\frac{1}{3}}$ and decays away from λ. One can also write an Airy type asymptotic near $|x| = \lambda$, but we do not need it here. The bounds of Lemma 5.1 follows from standard WKB bounds as in [4] and well-known formulas for the Hermite functions, see [1].

An immediate consequence of the above lemma is that all bounds in Theorem 4 are sharp in one dimension. In what follows we construct examples which show that the bounds in Theorem 4 are also sharp in higher dimension. For a positive integer N we consider eigenfunctions which correspond to the eigenvalue $n + 2N$. We define $\lambda > 0$ by

$$\lambda^2 = n + 2N$$

5.1. **Concentration in a tube in D_0.** We set $x' = (x_2, \ldots, x_n)$ and consider the eigenfunction

$$v(x) = \tilde{h}_N(x_1) e^{-\frac{1}{2}|x'|^2}$$

which is concentrated in the tube

$$T = \{|x_1| < \lambda, \qquad |x'| < 1\}$$

We have

$$\|v\|_{L^2} \sim 1$$

and, if $1 \le p \le \infty$

$$\|v\|_{L^p(T \cap D_0^{int})} \sim \lambda^{\frac{1}{p} - \frac{1}{2}}.$$

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5.2. **Point concentration in D_0.** We assume that N is even, otherwise the arguments need a small modification. We consider the set I of indices
$$I = \{\alpha \text{ even}; |\alpha| = N, \ \alpha_i > N/4n\}$$
Then we consider the eigenfunction
$$v = \sum_{\alpha \in I} \prod_{i=1}^{n} h_{\alpha_i}(x_i)$$
which concentrates in the ball $B(0, \lambda^{-1})$. Since
$$|I| \sim N^{n-1} \sim \lambda^{2(n-1)}$$
it follows that
$$\|v\|_{L^2} \sim \lambda^{n-1}$$
On the other hand, by Lemma 5.1, for $\alpha \in I$ we get
$$h_{\alpha_i}(x_i) \sim \lambda^{-\frac{1}{2}}, \qquad |x_i| < \lambda^{-1}$$
Summing up, we obtain
$$h(x) \sim |I|\lambda^{-\frac{n}{2}} \sim \lambda^{2(n-1)}\lambda^{2(n-1)}, \qquad x \in B(0, \lambda^{-1})$$
therefore
$$\|h\|_{L^p(B(0,\lambda^{-1}))} \sim \lambda^{-\frac{n}{p}}\lambda^{-n/2}\lambda^{2(n-1)}$$
Thus
$$\lambda^{1-n(\frac{1}{2}-\frac{1}{p})}\|v\|_{L^p(D_0^{int})} \gtrsim \|v\|_{L^2}$$
which one should compare to (10).

5.3. **Point concentration in D_j.** We fix r with $re_1 \in D_j^{int}$ and construct an eigenfunction which concentrates in $B(re_1, c2^j\lambda^{-1})$. Let N_0 be so that
$$|N_0 - \frac{\lambda^2 - r^2}{2}| < 1.$$
This implies that
$$|N_0| \sim 2^{-2j}\lambda^2$$
We consider the set of indices
$$I = \{\alpha| \ \alpha' \text{ even}; |\alpha| = N, \ \alpha_1 > N - N_0, \ \alpha_i' > N/4n\}$$
and its subset
$$J = \{\alpha \in I; \ h_{\alpha_1}(r) > \frac{1}{4}(\lambda^2 - r^2)^{-\frac{1}{2}}\}$$
Lemma 5.1 shows that near r the functions h_{α_1} oscillate at a frequency of the order of $2^{-j}\lambda$. Then, changing r by no more that $2^j\lambda^{-1}$, we can insure that $|J| \sim |I|$. We define
$$v = \sum_{\alpha \in J} \prod_{i=1}^{n} h_{\alpha_i}(x_i)$$
Since
$$|I| \sim N_0^{n-1}$$

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we have
$$\|v\|_{L^2} \sim |J|^{\frac{1}{2}} \sim 2^{-(n-1)j}\lambda^{n-1}$$
On the other hand, using Lemma 5.1 and the definition of J we compute
$$v(x) \sim 2^{\frac{j}{2}}\lambda^{-\frac{1}{2}}N_0^{-\frac{n-1}{4}}|J| \sim \lambda^{-\frac{n}{2}}2^{\frac{n-1}{2}j}2^{-2(n-1)j}\lambda^{2(n-1)}, \qquad x \in B(re_1, c2^j\lambda^{-1})$$
This gives
$$\|v\|_{L^p(B(re_1,c2^j\lambda^{-1}))} \sim 2^{\frac{nj}{p}}\lambda^{-\frac{n}{p}}\lambda^{-\frac{n}{2}}2^{\frac{n}{2}j}2^{-2(n-1)j}\lambda^{2(n-1)}$$
therefore
$$\lambda^{1-n(\frac{1}{2}-\frac{1}{p})}2^{j(1-n(\frac{1}{2}-\frac{1}{p}))}\|v\|_{L^p(D_j^{int})} \gtrsim \|v\|_{L^2}.$$

5.4. Concentration on a tube in D_j^{int}.
Here we construct eigenfunctions which concentrate in the tube
$$T = \{x \in D_j^{int};\ |x'| \le 2^{-\frac{j}{2}}\}$$
Now we consider the set of indices
$$I = \{\alpha;\ |\alpha| = N,\ \alpha'\ \text{even},\ |\alpha'_j - 2^j| \le c2^j\}$$
with small C. This has size
$$|I| \sim 2^{j(n-1)}$$
We want to consider a subset J of I of comparable size and set
$$v = \sum_{\alpha \in J}\prod_{i=1}^{n}h_{\alpha_i}(x_i)$$
The difficulty is that we want to avoid cancellations in this summation. For $\alpha \in I$ the functions $h_{\alpha'_i}(x'_i)$ are of size $2^{-j/4}$ in an interval of length $2^{-j/2}$ centered at zero. It remains to insure that the functions $h_{\alpha_1}(x_1)$ have the same sign within most of the tube. For this we need to check that their phases are coherent, i.e. they differ essentially by a constant.

Set $\mu = \sqrt{1 + \alpha_1}$. We must compare the functions $s_\mu^-(x)$ in the range
$$\lambda - |x| \sim 2^{-2j}\lambda, \qquad \lambda^2 - \mu^2 \sim 2^j$$
For this we compute
$$\frac{d}{d\mu}\frac{d}{dx}s_\mu^-(x) = \frac{2\lambda}{\sqrt{\lambda^2 - x^2}} \sim 2^j$$
Integrating this with respect to x and then with respect to μ we find that for x and μ_1, μ_2 as above we have
$$s_{\mu_1}^-(x) - s_{\mu_2}^-(x) = C(\mu_1, \mu_2) + O(2^{-j}\lambda|\mu_1 - \mu_2|)$$
which yields
$$|s_{\mu_1}^-(x) - s_{\mu_2}^-(x) - C(\mu_1, \mu_2)| \lesssim c$$
If c is small enough then the phases $s_\mu^-(x)$ are close modulo constants, therefore we can choose a subset of indices μ of comparable cardinality so that the phases

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are close modulo $2\pi\mathbb{Z}$. This leads to the subset J of I corresponding to this restricted set of α_1's.

Now we can complete the computation. On one hand

$$\|v\|_{L^2} \sim 2^{\frac{n-1}{2}j}$$

On the other hand, for most $x \in T$ we have

$$|v(x)| \sim 2^{\frac{j}{2}}\lambda^{-\frac{1}{2}}2^{-\frac{n-1}{4}j}|J| \sim 2^{(n-1)j}\lambda^{-\frac{1}{2}}2^{-\frac{n-3}{4}j}$$

Since $|T| \sim 2^{-\frac{n+3}{2}j}\lambda$ this yields

$$\|v\|_{L^p(T)} \sim 2^{-\frac{n+3}{2p}j}\lambda^{\frac{1}{p}}2^{(n-1)j}\lambda^{-\frac{1}{2}}2^{-\frac{n-3}{4}j}$$

Hence

$$\|u\|_{L^2} \lesssim \lambda^{\frac{1}{2}-\frac{1}{p}}2^{-\frac{j}{2}\frac{n+1}{2}}2^{\frac{j}{2}\frac{n+3}{p}}\|u\|_{L^p(D_j^{int})}.$$

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