Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation

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Abstract

The aim of this article is threefold. First, we use the FBI transform to set up a calculus for partial differential operators with non-smooth coefficients. Next, this calculus allows us to prove Strichartz type estimates for the wave equation with nonsmooth coefficients. Finally, we use these Strichartz estimates to improve the local theory for second order nonlinear hyperbolic equations.

1 Introduction

The FBI transform is, in a way, similar to the complex Fourier transform, in that for each function in R^n it provides a representation as a holomorphic function in R^{2n}. However, in the case of the FBI transform we can identify naturally R^{2n} with the phase space T^*R^n.

For a pseudodifferential operator with smooth symbol acting on functions in R^n one can produce by conjugation a corresponding formal series acting on functions in R^{2n}, for which the first term is exactly the multiplication by the symbol. This series converges and has a nice representation in the Weyl calculus provided that the symbol of the operator is analytic. This is how the FBI transform has been used in the study of partial differential operators with analytic coefficients; see [12], [13], where this machinery is developed.

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Here, in a way, we do the opposite: we look at operators with nonsmooth coefficients, approximate the conjugated operator by a partial sum of the formal series, and then we prove error estimates. In the simplest case the approximate conjugate operator is exactly the multiplication by the symbol. This is also related to the Cordoba-Fefferman wave-packet transform in [3].

In the third section we use the error estimates to reduce the Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients to weighted $L^p(L^q) \to L^2$ estimates for the FBI transform. These, in turn can be proved in the usual fashion, using appropriate oscillatory integral estimates.

In the last part of the article we explain how one can use the new Strichartz type estimates to improve the local theory for nonlinear hyperbolic equations beyond the classical setup. These results are not sharp and will be improved in subsequent articles.

2 A calculus for operators with nonsmooth coefficients

The calculus we develop is dependent on the frequency; thus, in order to use it for general pseudodifferential operators one needs to start with a Paley-Littlewood decomposition and then use the calculus for each dyadic piece separately. The parameter λ below represents the size of the frequency.

The FBI transform of a temperate distribution f is a holomorphic function in C^n defined as

$$(T_\lambda f)(z) = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int e^{-\frac{\lambda}{2}(z-y)^2} f(y)\, dy \tag{2.1}$$

To understand better how the FBI transform works, consider the L^2 normalized function

$$f_{x_0,\xi_0}(y) = \lambda^{\frac{n}{4}} \pi^{-\frac{n}{4}} e^{-\frac{\lambda}{2}(y-x_0)^2} e^{i\lambda(y-x_0)}$$

which is localized in a $\lambda^{-\frac{1}{2}}$ neighborhood of x_0 and frequency localized in a $\lambda^{\frac{1}{2}}$ neighborhood of $\lambda\xi_0$. Due to the uncertainty principle this is the best one can do when trying to localize in both space and frequency. Then

$$(T_\lambda f)(z) = \lambda^{-\frac{n}{4}} \pi^{\frac{n}{4}} e^{\frac{\lambda}{4}(z-x_0+i\xi_0)^2 - \frac{\lambda}{2}(z-x_0)^2} = \lambda^{-\frac{n}{4}} \pi^{\frac{n}{4}} e^{-\frac{\lambda}{4}|z-x_0+i\xi_0|^2} e^{\frac{\lambda}{2}|\Im z|^2} e^{-i\frac{\lambda}{2}(\Re z - x_0)(\Im z - \xi_0)}$$

Modulo the common factor $e^{\frac{\lambda}{2}|\Im z|^2}$ this is localized in a $\lambda^{-\frac{1}{2}}$ neighborhood of $x_0 - i\xi_0$. Hence, it is natural to introduce the notation

$$z = x - i\xi.$$

Like the Fourier transform, the FBI transform has good L^2 properties. Set

$$\Phi(z) = e^{-\lambda\xi^2}$$

Then the operator T_λ is an isometry from $L^2(R^n)$ onto the closed subspace of holomorphic functions in $L^2_\Phi(C^n)$. One inversion formula is provided by the adjoint operator:

$$f(y) = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int \Phi(z) e^{-\frac{\lambda}{2}(\bar{z}-y)^2} (T_\lambda f)(z)\ dx d\xi$$

This is of course not the only possible inversion formula since the range of T_λ consists only of holomorphic functions.

Let $a(x,\xi)$ be a compactly supported symbol. Then

$$a_\lambda(x,\xi) = a(x, \frac{\xi}{\lambda})$$

is a symbol supported at frequency λ.

What we want is to determine the conjugate $\tilde{A}_\lambda$ of $A_\lambda(y,D)$ with respect to F_λ,

$$T_\lambda A_\lambda(y,D) \approx \tilde{A}_\lambda T_\lambda$$

modulo a small remainder. Start with some simple symbols. For the conjugate of x compute

$$T_\lambda(yf)(z) = (x + \frac{1}{-i\lambda}(\partial_\xi - \lambda\xi))T_\lambda f$$

The conjugate of $\frac{D}{\lambda}$ is of course $\frac{D_x}{\lambda}$, but we shall write it as

$$T_\lambda(\frac{D}{\lambda} f)(z) = (\xi + \frac{1}{\lambda}(\frac{1}{i}\partial_x - \lambda\xi))T_\lambda f$$

Based on this, we get the formal asymptotics

$$T_\lambda A_\lambda(x,D) \approx \sum_{\alpha,\beta} (\partial_\xi - \lambda\xi)^\alpha \frac{\partial_x^\alpha \partial_\xi^\beta a(x,\xi)}{\alpha!\beta!(-i\lambda)^{|\alpha|}\lambda^|\beta|} (\frac{1}{i}\partial_x - \lambda\xi)^\beta T_\lambda$$

Now we want to make these asymptotics rigorous. Given $s > 0$ define the partial sum

$$\tilde{a}^s_\lambda = \sum_{|\alpha|+|\beta|<s} (\partial_\xi - \lambda\xi)^\alpha \frac{\partial_x^\alpha \partial_\xi^\beta a(x,\xi)}{\alpha!\beta!(-i\lambda)^{|\alpha|}\lambda^{|\beta|}} (\frac{1}{i}\partial_x - \lambda\xi)^\beta$$

For instance

$$\tilde{a}^s_\lambda = a \qquad s \le 1 \tag{2.2}$$

If $1 < s \leq 2$ then

$$\tilde{a}_\lambda^s = a + \frac{1}{-i\lambda} a_x(\partial_\xi - \lambda\xi) + \frac{1}{\lambda} a_\xi(\frac{1}{i}\partial_x - \lambda\xi)$$

Since we only consider this operator on holomorphic functions, we can also rewrite it in a complex fashion as

$$\tilde{a}_\lambda^s = a + \frac{2}{\lambda}(\bar{\partial} a)(\partial - i\lambda\xi), \qquad 1 < s \leq 2 \tag{2.3}$$

Define the remainder

$$R_{\lambda,a}^s = T_\lambda A_\lambda - \tilde{a}_\lambda^s T_\lambda$$

Then our main result is

Theorem 1 *Assume that $a \in C_x^s(C_0^\infty)$. Then*

$$\|R_{\lambda,a}^s\|_{L^2 \to L^2_\phi} \leq c\lambda^{-\frac{s}{2}} \tag{2.4}$$

In other words, this theorem shows that the order s approximation is precise up to $s/2$ derivatives.

Proof: We start with a preliminary result, which in particular shows that the α term in the sum is of the order of $\lambda^{-\frac{\alpha}{2}}$:

Lemma 2.1 *If $u \in L^2_\phi$ is holomorphic then*

$$\|(\partial_\xi - \lambda\xi)^\alpha u\|_{L^2_\Phi} = c_\alpha \lambda^{\frac{|\alpha|}{2}} \|u\|_{L^2_\Phi}$$

$$\|(\partial_x - i\lambda\xi)^\alpha u\|_{L^2_\Phi} = c_\alpha \lambda^{\frac{|\alpha|}{2}} \|u\|_{L^2_\Phi}$$

Proof: Represent u as $u = T_\lambda f$ with $f \in L^2$. Then

$$(\partial_\xi - \lambda\xi)^\alpha u = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int (x-y)^\alpha e^{-\frac{\lambda}{2}(z-y)^2} f(y) dy$$

Hence

$$\begin{aligned}
& \|(\partial_\xi - \lambda\xi)^\alpha u\|^2_{L^2_\Phi} = \\
= {} & \lambda^{\frac{3n}{2}} 2^{-n} \pi^{-\frac{3n}{2}} \int (x-y)^\alpha e^{-\frac{\lambda}{2}(z-y)^2} f(y)(x-w)^\alpha e^{-\frac{\lambda}{2}(\bar{z}-w)^2} f(w) e^{-\lambda\xi^2} \, dx d\xi dy dw \\
= {} & \lambda^{\frac{3n}{2}} 2^{-n} \pi^{-\frac{3n}{2}} \int e^{-\frac{\lambda}{2}[(x-y)^2+(x-w)^2]} (x-y)^\alpha f(y)(x-w)^\alpha f(w) e^{i\lambda\xi(y-w)} \, dx d\xi dy dw \\
= {} & \lambda^{\frac{n}{2}} \pi^{-\frac{n}{2}} \int (x-y)^{2\alpha} e^{-\lambda(y-x)^2} |f(y)|^2 dy dx \\
= {} & c_\alpha \lambda^{|\alpha|} \|f\|^2_{L^2} = c_\alpha \lambda^{|\alpha|} \|u\|^2_{L^2_\Phi}
\end{aligned}$$

For the second part we express $T_\lambda f$ in terms of the Fourier transform of f,

$$\begin{aligned} T_\lambda f(z) &= \lambda^{\frac{n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int e^{-\frac{1}{2\lambda}\eta^2} e^{iz\eta} \hat{f}(\eta) d\eta \\ &= \lambda^{\frac{5n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int e^{-\frac{\lambda}{2}\eta^2} e^{i\lambda z\eta} \hat{f}(\lambda\eta) d\eta \end{aligned}$$

therefore

$$T_\lambda f(z) = \lambda^{\frac{5n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} e^{\frac{\lambda}{2}\xi^2} \int e^{-\frac{\lambda}{2}(\eta-\xi)^2} e^{i\lambda x\eta} \hat{f}(\lambda\eta) d\eta \tag{2.5}$$

Then

$$\begin{aligned} &\|(\frac{1}{i}\partial_x - \lambda\xi)^\alpha u\|^2_{L^2_\Phi} = \\ &= \lambda^{\frac{5n}{2}} 2^{-n} \pi^{-\frac{3n}{2}} \int (\eta-\xi)^\alpha (\mu-\xi)^\alpha e^{-\frac{\lambda}{2}[(\eta-\xi)^2+(\mu-\xi)]^2} e^{i\lambda x(\eta-\mu)} \hat{f}(\lambda\eta)\hat{f}(\lambda\mu) d\eta d\mu dx d\xi \\ &= \lambda^{\frac{3n}{2}} \pi^{-\frac{n}{2}} \int (\eta-\xi)^{2\alpha} e^{-\lambda(\eta-\xi)^2} |\hat{f}(\lambda\eta)|^2 d\eta d\xi \\ &= c_\alpha \lambda^\alpha \|f\|^2_{L^2} = c_\alpha \lambda^{|\alpha|} \|u\|^2_{L^2_\Phi} \end{aligned}$$

q.e.d.

To prove the theorem, observe that it suffices to prove it for symbols of the form $b(x)c(\xi)$ with $b \in C^s$ and c smooth. This reduction can be achieved for instance if we replace a with its Fourier series in ξ with respect to a larger set containing the support of a and then truncate the functions in the series near the support of a in ξ (see also [20], pp 37).

For b we can make the same computation as in Lemma 2.1 to get

$$(R^s_{\lambda,b} T_\lambda f)(z) = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int k(x,y) e^{-\frac{\lambda}{2}(z-y)^2} f(y) dy$$

where

$$k(x,y) = b(y) - \sum_{|\alpha|<s} \frac{b^{(\alpha)}(x)}{\alpha!}(x-y)^\alpha$$

Then

$$|k(x,y)| \le c|x-y|^s \tag{2.6}$$

therefore, computing as in Lemma 2.1 we get

$$\|R^s_{\lambda,b} T_\lambda f\|^2_{L^2_\Phi} = \lambda^{\frac{n}{2}} \pi^{\frac{-n}{2}} \int k^2(x,y) e^{-\lambda(x-y)^2} f^2(y) dx dy$$

By (2.6) the integral in x is of the order of $\lambda^{-\frac{n}{2}-s}$ therefore

$$\|R^s_{\lambda,b}\|_{L^2 \to L^2_\phi} \le c\lambda^{-\frac{s}{2}} \tag{2.7}$$

To prove the result for c observe that, as in the second part of Lemma 2.1,

$$(R^s_{\lambda,c}f)(z) = \lambda^{\frac{5n}{4}} 2^{\frac{n}{2}} \pi^{-\frac{3n}{4}} e^{\frac{\lambda}{2}\xi^2} \int h(\xi,\eta) e^{-\frac{\lambda}{2}(\eta-\xi)^2} e^{i\lambda x\eta} \hat{f}(\lambda\eta) d\eta \tag{2.8}$$

where

$$h(\eta,\xi) = c(\eta) - \sum_{|\alpha|<s} \frac{c^{(\alpha)}(\xi)}{\alpha!}(\eta-\xi)^\alpha$$

Hence

$$|h(\xi,\eta)| \leq c|\xi-\eta|^s \tag{2.9}$$

Computing as in Lemma 2.1 we get

$$\|R^s_{\lambda,c}f\|^2_{L^2_\Phi} = \lambda^{\frac{3n}{2}} \pi^{-\frac{n}{2}} \int h^2(\xi,\eta) e^{-\frac{\lambda}{2}(\xi-\eta)^2} \hat{f}^2(\lambda\eta) d\xi d\eta$$

The integral in ξ is of the order of $\lambda^{-\frac{n}{2}-s}$ therefore

$$\|R^s_{\lambda,c}\|_{L^2\to L^2_\phi} \leq c\lambda^{-\frac{s}{2}}$$

Before moving on to $a(x,\xi) = b(x)c(\xi)$ we need to prove a stronger version of the above estimate, namely that

$$\|\frac{(\partial_\xi - \lambda\xi)^\alpha}{\lambda^\alpha} R^s_{\lambda,c}\|_{L^2\to L^2_\Phi} \leq \lambda^{-\frac{s+|\alpha|}{2}} \qquad |\alpha| < s \tag{2.10}$$

From (2.8) we get

$$\frac{(\partial_\xi - \lambda\xi)^\alpha}{\lambda^\alpha} R^s_{\lambda,c}f = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} e^{\frac{\lambda}{2}\xi^2} \int k(\xi,\eta) e^{-\frac{\lambda}{2}(\eta-\xi)^2} e^{i\lambda x\eta} \hat{f}(\lambda\eta) d\eta$$

where

$$k(\xi,\eta) = (\lambda^{-1}\partial_\xi - (\xi-\eta))^\alpha h(\xi,\eta).$$

Then

$$|k(\xi,\eta)| \leq |\xi-\eta|^{|\alpha|+s} + \lambda^{-\frac{|\alpha|+s}{2}}$$

therefore (2.10) follows in the same manner as (2.7).

Now prove the result for $a = b(x)c(\xi)$. We have

$$\begin{aligned} T_\lambda A_\lambda &= T_\lambda b C_\lambda \\ &= \tilde{b}^s_\lambda T_\lambda C_\lambda + R^s_{\lambda,b} C_\lambda \\ &= \tilde{b}^s_\lambda \tilde{c}^s_\lambda T_\lambda + \tilde{b}^s_\lambda R^s_{\lambda,c} + R^s_{\lambda,b} C_\lambda \end{aligned}$$

The last two terms are $O(\lambda^{-\frac{s}{2}})$ due to (2.10), respectively (2.7). The difference $\tilde{a}^s_\lambda - \tilde{b}^s_\lambda \tilde{c}^s_\lambda$, on the other hand, consists of terms which contain at least s factors of the form

$$\lambda^{-1}(\partial_\xi - \lambda\xi), \qquad \lambda^{-1}(\frac{1}{i}\partial_x - \lambda\xi)$$

But these operators are equal on holomorphic functions and commuting them yields a λ^{-1} factor, therefore the difference can be estimated using Lemma 2.1.

Remark 2.2 *In a similar manner one can prove the following analogue of (2.10),*

$$\| \frac{(\partial_\xi - \lambda\xi)^\alpha}{\lambda^\alpha} R^s_{\lambda,b} \|_{L^2 \to L^2_\Phi} \leq \lambda^{-\frac{s+|\alpha|}{2}} \qquad |\alpha| < s \tag{2.11}$$

Combined with (2.10) this yields a generalization of (2.4), namely

$$\| \frac{(\partial_\xi - \lambda\xi)^\alpha}{\lambda^\alpha} R^s_{\lambda,a} \|_{L^2 \to L^2_\Phi} \leq c_\alpha \lambda^{-\frac{s+|\alpha|}{2}} \qquad |\alpha| < s \tag{2.12}$$

For the study of nonlinear hyperbolic equations it is useful to produce a version of the above results which corresponds to partial differential operators with coefficients in mixed L^p spaces. Denote by $x = (x_0, x')$ the coordinates in $R \times R^n$. In the sequel we use the notation $L^p(L^q)$ for $L^p_{x_0}(L^q_{x'})$. For simplicity we prove exactly the estimate we shall use later on. The reader can easily repeat the same argument for a different choice of spaces. Set $X = L^2(L^\infty)$ and

$$X^1 = \{u \in X;\ \nabla u \in X\}$$

Then

Theorem 2 *Assume that $a \in X^1(C_0^\infty)$. Then*

$$\|T_\lambda A_\lambda - aT_\lambda\|_{L^\infty(L^2) \to L^2_\Phi} \leq c\lambda^{-\frac{1}{2}}$$

Proof: As before the problem reduces to the case when $a(x,\xi) = b(x)c(\xi)$ with $b \in X^1$ and $c \in C_0^\infty$. The L^2 estimate for c is already proved. Let us prove the estimate for b. Arguing as in the previous theorem,

$$R^1_{\lambda,b} f(z) = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{2}} \int (b(x) - b(y)) e^{-\frac{\lambda}{2}(z-y)^2} f(y) dy$$

therefore

$$\|R^1_{\lambda,b} f\|^2_{L^2_\Phi} = \lambda^{\frac{n}{2}} \pi^{-\frac{n}{2}} \int |b(x) - b(y)|^2 e^{-\frac{\lambda}{2}(x-y)^2} f^2(y) dx dy$$

Since f is in $L^\infty(L^2)$, we need to obtain an $L^1(L^\infty)$ bound for the multiplier

$$K(y) = \lambda^{\frac{n}{2}} \int |b(x) - b(y)|^2 e^{-\lambda(x-y)^2} dx$$

Rewrite it as

$$K(y) = \int_0^1 \int_0^1 \lambda^{\frac{n}{2}} \int < \nabla b(y + hx), x >< \nabla b(y + kx), x > e^{-\lambda x^2} dxdhdk.$$

Since $\nabla b \in L^2(L^\infty)$, this implies that

$$\|K\|_{L^1(L^\infty)} \leq c\lambda^{-1} \|b\|^2_{X^1}$$

which leads to the remainder bound

$$\|R^1_{\lambda,b}\|_{L^\infty(L^2) \to L^2_\Phi} \leq \lambda^{-\frac{1}{2}}$$

Then we can compute

$$R^1_{\lambda,a} = R^1_{\lambda,c} b + cR^1_{\lambda,b} + T_\lambda[b, C(\frac{D}{\lambda})]$$

The first two right hand side terms have an $L^\infty(L^2) \to L^2_\Phi$ norm of $O(\lambda^{-\frac{1}{2}})$ due to the error estimates for b and c. Since $\nabla b \in L^2(L^\infty)$, the commutator $[b, C(\frac{D}{\lambda})]$ has an $O(\lambda^{-1})$ norm:

Lemma 2.3 *Let $b \in X^1$ and $c \in C_0^\infty$. Then the following commutator estimate holds:*

$$\|[b, C_\lambda]\|_{L^\infty(L^2) \to L^2} \leq c\lambda^{-1} \tag{2.13}$$

Proof: The kernel of $[b, C_\lambda]$ is

$$\begin{aligned} K(x, y) &= \int (b(x) - b(y))c(\frac{\xi}{\lambda}) e^{i(x-y)\xi} d\xi \\ &= \lambda^n (b(x) - b(y))\hat{c}(\lambda(x - y)) \\ &= \lambda^n \int_0^1 < \nabla b(x + t(y - x)), y - x > \hat{c}(\lambda(x - y))dt \end{aligned}$$

Then

$$\begin{aligned} [b, C_\lambda]f(x) &= \lambda^n \int \int_0^1 < \nabla b(x + t(y - x)), y - x > \hat{c}(\lambda(x - y))f(y)dtdy \\ &= \lambda^n \int \int_0^1 < \nabla b(x + tw), w > \hat{c}(\lambda w)f(x + w)dtdw \end{aligned}$$

therefore

$$\begin{aligned} \|[b, C_\lambda]f\|_{L^2} &\leq c \int \lambda^n |w||\hat{c}(\lambda w)|dw \|f\|_{L^\infty(L^2)} \|\nabla b\|_{L^2(L^\infty)} \\ &\leq c\lambda^{-1} \|\nabla b\|_{L^2(L^\infty)} \end{aligned}$$

q.e.d.

3 Strichartz estimates for the wave equation with non-smooth coefficients

The Strichartz estimates are $L^p(L^q)$ estimates for solutions to the wave equation. These estimates have been very useful in the study of semilinear hyperbolic equations. One form of the estimates applies to solutions to the homogeneous wave equation,

$$\Box u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1$$

Then

$$\|u\|_{L^p(L^q)} \leq c\|u_0\|_{H^\rho} + \|u_1\|_{H^{\rho-1}} \tag{3.14}$$

provided that $2 \leq p \leq \infty$, $2 \leq q \leq \infty$ and

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \rho, \qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \tag{3.15}$$

with the sole exception of the pair $(1, 2, \infty)$ in dimension $n = 3$.

In the sequel, call Strichartz pairs all the triplets (ρ, p, q) satisfying the above relations except for the forbidden endpoint $(1, 2, \infty)$ in dimension $n = 3$. If the equality holds in the second relation,

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \rho, \qquad \frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \tag{3.16}$$

then we call (ρ, p, q) a sharp Strichartz pair. The estimates for any Strichartz pair follow by Sobolev embeddings from the estimates for sharp Strichartz pairs.

A special role is played in dimension $n \geq 4$ by the sharp Strichartz pair $(\frac{n+1}{2(n-1)}, 2, \frac{2(n-1)}{n-3})$ which we call the endpoint. Then all Strichartz estimates can be recovered from the endpoint estimate and the energy estimate (which corresponds to $(0, \infty, 2)$) by interpolation and Sobolev embeddings. The 3-dimensional correspondent is the forbidden endpoint $(1, 2, \infty)$.

The second form of the estimates applies to solutions to the inhomogeneous wave equation,

$$\Box u = f, \quad u(0) = 0, \quad u_t(0) = 0$$

Then

$$\|D^{1-\rho-\rho_1} u\|_{L^p(L^q)} \leq \|f\|_{L^{p'_1}(L^{q'_1})} \tag{3.17}$$

for all Strichartz pairs (ρ, p, q), (ρ_1, p_1, q_1).

Estimates of this type were first obtained in [2], [17]. Further references can be found in a more recent expository article [4]. The endpoint estimate was only recently proved in [7] $(n \geq 4)$.

Consider now a variable coefficient second order hyperbolic equation

$$P(x,D)u = 0, \qquad u(0) = u_0, \quad u_t(0) = u_1 \tag{3.18}$$

where

$$P(x,D) = -\partial_i g^{ij}(x)\partial_j$$

If the coefficients g^{ij} are smooth then the estimates hold locally, see [11] (except for the endpoint). For time independent $C^{1,1}$ coefficients, in dimension $n = 2,3$, the estimates are proved in [14]. Furthermore, in [15] they are shown to fail for C^s coefficients, $s < 2$.

Here we prove that some weaker Strichartz estimates hold if the coefficients are C^s, $0 < s \leq 1$. Of course, if $s < 1$ then the hyperbolic problem is not necessarily L^2 well-posed, therefore our estimates are just a-priori bounds. In the sequel we assume that the matrix of the coefficients g^{ij} and its inverse g_{ij} are uniformly bounded,

$$\sum_{i,j} |g^{ij}| + |g_{ij}| \leq C$$

Theorem 3 *Assume that P is in divergence form and has C^s coefficients, $0 \leq s \leq 1$. Let (ρ, p, q) be a Strichartz pair. Then*

$$\|D^{1-\rho+\frac{s-2}{4}} u\|_{L^p(L^q)} \leq (1 + \|g\|_{C^s})\|u\|_{H^1} + \|Pu\|_{H^{\frac{s-2}{2}}} \tag{3.19}$$

whenever the right hand side is finite and u is supported in a fixed compact set.

Thus, compared to the sharp estimates in (3.14), there is a loss of $(s-2)/4$ derivatives. This is consistent with the earlier results which suggest that the sharp estimates should hold for operators with C^2 coefficients. In effect this result is what one would get by formally interpolating a sharp estimate for C^2 coefficients, with Sobolev embeddings for C^0 coefficients. The case $1 < s \leq 2$ is considered in a forthcoming paper of the author [19].

Proof: Let $s(\xi)$ be a smooth symbol supported in $\frac{1}{2} \leq |\xi| \leq 2$ so that

$$\sum_j s(2^{-j}\xi) = 1$$

Let $a(\xi)$ be a smooth, compactly supported cutoff function which is 1 in the region

$$U = \{\frac{1}{4} \leq |\xi| \leq 4\}$$

and is supported away from 0.

Given $\lambda > 1$ define

$$S_\lambda = S(\frac{D}{\lambda}), \qquad A_\lambda = A(\frac{D}{\lambda})$$

Now we do a Paley-Littlewood decomposition and reduce (3.19) to corresponding dyadic estimates for $S_\lambda u$,

$$\lambda^{1-\rho+\frac{s-2}{4}} \|S_\lambda u\|_{L^p(L^q)} \leq \lambda(1 + \|g\|_{C^s})\|S_\lambda u\|_{L^2} + \lambda^{\frac{s-2}{2}} \|A_\lambda P S_\lambda u\|_{L^2} \tag{3.20}$$

To get (3.19) from (3.20) it suffices to sum the square of (3.20) and get a good L^2 estimate for the difference

$$A_\lambda P S_\lambda - S_\lambda P_\lambda = A_\lambda[P, S_\lambda]$$

We claim that

$$\lambda^{\frac{s-2}{2}} \|A_\lambda[P, S_\lambda]u\|_{L^2} \leq \lambda^{-\frac{s}{2}} \|g\|_{C^s} \|u\|_{H^1}$$

which is more than sufficient for the summation. Since P is in divergence form, this follows from the corresponding commutator estimate for the coefficients,

$$\|[g, S_\lambda]\|_{L^2 \to L^2} \leq \lambda^{-s} \tag{3.21}$$

Indeed, the kernel of $[g, S_\lambda]$ is

$$H(x, y) = \lambda^n (g(x) - g(y)) \hat{s}(\lambda(x - y))$$

Then

$$H(x, y) \leq \lambda^{n-s}(1 + \lambda|x - y|)^{-N}$$

and the L^2 bound (3.21) follows.

Now use the FBI transform and set

$$v_\lambda = T_\lambda S_\lambda u$$

Observe first that v_λ is concentrated in the region U defined before. Indeed, outside this region we have

$$\|v_\lambda\|_{L^2_\Phi(U^c)} \leq e^{-c\lambda} \|S_\lambda u\|_{L^2}$$

which is a straightforward consequence of the representation formula (2.5).

It remains to estimate v_λ inside this region. Let $a(\xi)$ be a smooth, compactly supported cutoff function which is 1 in U and is supported away from 0. Then write

$$A_\lambda P S_\lambda = \lambda^2 A_\lambda \frac{D_i}{\lambda} g^{ij}(x) \frac{D_j}{\lambda} A_\lambda S_\lambda$$

If we apply Theorem 1 succesively for the operators $A_\lambda \frac{D_i}{\lambda}$ and $g^{ij}(x)\frac{D_j}{\lambda}A_\lambda$ then we obtain

$$\|T_\lambda A_\lambda P S_\lambda u - \lambda^2 a^2(\xi) p(x,\xi) T_\lambda S_\lambda u\|_{L^2_\Phi} \leq \lambda^{2-\frac{s}{2}} \|g\|_{C^s} \|S_\lambda u\|_{L^2}$$

and further

$$\lambda^{\frac{s+2}{2}} \|a^2(\xi) p(x,\xi) v_\lambda\|_{L^2_\Phi} \leq \lambda^{\frac{s-2}{2}} \|A_\lambda P S_\lambda u\|_{L^2} + \lambda \|g\|_{C^s} \|S_\lambda u\|_{L^2}$$

On the other hand we have the trivial L^2 estimate,

$$\lambda \|v_\lambda\|_{L^2} = \lambda \|S_\lambda u\|_{L^2}$$

Since $S_\lambda u = T_\lambda^* v_\lambda$, the last two estimates show that (3.20) would follow from the following inequality:

$$\lambda^{1-\rho+\frac{s-2}{4}} \|T_\lambda^* v_\lambda\|_{L^p(L^q)} \leq \lambda \|v_\lambda\|_{L^2} + \lambda^{\frac{s+2}{2}} \|p(x,\xi) v_\lambda\|_{L^2}$$

for v_λ supported in U.

Note that from this point further we no longer use the regularity of the coefficients. All that matters is that the coefficients are bounded and that the quadratic form $p(x,\xi)$ is non-degenerate uniformly in x.

The last inequality is equivalent to a uniform bound for

$$U_\lambda = \lambda^{-\frac{1}{2}-\rho} T_\lambda^* \frac{a(\xi)}{\lambda^{-\frac{s}{4}} + \lambda^{\frac{s}{4}} |p(x,\xi)|} : L^2_\phi \to L^p(L^q)$$

and further to the corresponding bound for $U_\lambda \Phi U_\lambda^*$,

$$\lambda^{-1-2\rho} T_\lambda^* \frac{a^2(\xi)\Phi(\xi)}{(\lambda^{-\frac{s}{4}} + \lambda^{\frac{s}{4}} |p(x,\xi)|)^2} T_\lambda : L^{p'}(L^{q'}) \to L^p(L^q)$$

The weight inside is integrable across the level sets of p. Hence, foliating with respect to the level sets of p this reduces to

$$\lambda^{-1-2\rho} T_\lambda^* a^2(\xi) \Phi(\xi) \delta_{p(x,\xi)=0} T_\lambda : L^{p'}(L^{q'}) \to L^p(L^q)$$

(One should in effect work with all level sets and use $\delta_{p(x,\xi)=0}$, but this makes no difference). Thus, if we set

$$V_\lambda = T_\lambda^* a^2(\xi) \Phi(\xi) \delta_{p(x,\xi)=0} T_\lambda$$

then we need to prove the uniform estimate

$$\|V_\lambda\|_{L^{p'}(L^{q'}) \to L^p(L^q)} \leq \lambda^{1+2\rho} \tag{3.22}$$

We prove this when

(i) $(\rho, p, q) = (0, \infty, 2)$ (the energy estimate).

(ii) (ρ, p, q) is a sharp Strichartz pair with $2 < p \leq q$.

(iii) $(\rho, p, q) = (\frac{n+1}{2(n-1)}, 2, \frac{2(n-1)}{n-3})$, $n \geq 4$ (the endpoint estimate).

The other cases will follow by interpolation and Sobolev embeddings.

Remark 3.1 *Observe that V_λ can also be expressed in the form $V_\lambda = W_\lambda \Phi W_\lambda^*$, where*

$$W_\lambda f = T_\lambda f \delta_{p(x,\xi)=0}, \quad W_\lambda \Phi : L^2_{\Phi\delta_{p(x,\xi)=0}} \to L^p(L^q)$$

Then the estimate(3.22) is equivalent to the corresponding bound for W_λ,

$$\|W_\lambda \Phi\|_{L^2_{\Phi\delta_{p(x,\xi)=0}} \to L^p(L^q)} \leq \lambda^{\frac{1}{2}+\rho} \tag{3.23}$$

and to the dual bound,

$$\|W_\lambda^*\|_{L^{p'}(L^{q'}) \to L^2_{\Phi\delta_{p(x,\xi)=0}}} \leq \lambda^{\frac{1}{2}+\rho}$$

(i) The "energy" estimate. In the case when $(\rho, p, q) = (0, \infty, 2)$ we can factor out the FBI transform with respect to the y' variable. The remaining operator is

$$\lambda^{-1} T_{\lambda,0}^* a^2(\xi) \Phi(\xi_0) \delta_{p(x,\xi)=0} T_{\lambda,0}$$

where $T_{\lambda,0}$ is the FBI transform with respect to x_0. This is a multiplication operator with respect to the (x', ξ') variables. To prove that it is bounded from $L^1(L^2)$ into $L^\infty(L^2)$ we only need to verify that its kernel $K(y_0, \tilde{y}_0, x', \xi')$ with respect to the y_0 variable is bounded. We have

$$K(y_0, \tilde{y}_0, x', \xi') = \lambda^{-1} \lambda^{\frac{3}{2}} \int e^{-\frac{\lambda}{2}(x_0 - y_0)^2} e^{-\frac{\lambda}{2}(x_0 - \tilde{y}_0)^2} e^{i\lambda\xi_0(y_0 - \tilde{y}_0)} \delta_{p(x,\xi)=0} a^2(\xi) dx_0 d\xi_0$$

Because of the hyperbolicity condition , $p_{\xi_0} \neq 0$ on $p = 0$. Hence the integral with respect to ξ_0 is bounded and the conclusion easily follows.

(ii) The non-endpoint estimate. To prove (3.22) in the case when $2 < p \leq q$ we use interpolation. Consider the analytic family of operators

$$V_\lambda^\theta = \lambda^{-\theta(2\rho+1)} (\theta - 1) e^{\theta^2} T_\lambda^* a^2(\xi) \Phi(\xi) p^{-\theta}(x, \xi) T_\lambda$$

so that $\lambda^{-1-2\rho} V_\lambda = V_\lambda^1$. Then our estimate follows by interpolation from the following:

$$V_\lambda^\theta : L^2 \to L^2 \qquad \Re\theta = 0$$

$$V_\lambda^\theta : L^{p_1'}(L^1) \to L^{p_1}(L^\infty) \qquad \Re\theta = \theta_1$$

where p_1, θ_1 are chosen so that the points

$$(\frac{1}{2}, \frac{1}{2}, 0), \qquad (\frac{1}{p}, \frac{1}{q}, 1), \qquad (\frac{1}{p_1}, 0, \theta_1) \tag{3.24}$$

are collinear.

The L^2 bound follows from the L^2 boundedness of the FBI transform. For the remaining $L^{p_1'}(L^1) \to L^{p_1}(L^\infty)$ bound we estimate the kernel K_θ of V_λ^θ, which is given by

$$K_\theta(y,\tilde{y}) = \lambda^{\frac{3(n+1)}{2}-\theta(2\rho+1)} \int e^{i\lambda(y-\tilde{y})\xi} e^{-\frac{\lambda}{2}(y-x)^2} e^{-\frac{\lambda}{2}(\tilde{y}-x)^2} \Gamma(-\theta) e^{\theta^2} p^{-\theta}(x,\xi) a^2(\xi) dx d\xi \tag{3.25}$$

First we estimate the integral with respect to ξ in (3.25) using the standard oscillatory integral estimates

$$\int e^{iy\xi} a(\xi) p(\xi)^{-\theta} d\xi \le c(1+|y|)^{(\Re\theta-1)-\frac{n-1}{2}} \tag{3.26}$$

where p is a non-degenerate quadratic form and $a \in C_0^\infty$ is a smooth function supported away from 0. What is important in these estimates is that the characteristic cone $K = \{p = 0\}$ has $n-1$ nonvanishing curvatures (see e.g.Stein [16] 8.3.1 and 9.1.2).

If we use (3.26) to estimate the ξ integral in (3.25) then for $\Re\theta = \theta_1$ we obtain

$$\begin{aligned} |K_\theta(y,\tilde{y})| &\le \int \lambda^{\frac{3(n+1)}{2}-\Re\theta(2\rho+1)} (1+\lambda|y-\tilde{y}|)^{(\Re\theta-1)-\frac{n-1}{2}} e^{-\frac{\lambda}{2}(y-x)^2} e^{-\frac{\lambda}{2}(\tilde{y}-x)^2} dx \\ &\le \lambda^{n+1-\Re\theta(2\rho+1)} (1+\lambda|y-\tilde{y}|)^{(\Re\theta-1)-\frac{n-1}{2}} e^{-\frac{\lambda}{4}(y-\tilde{y})^2} \end{aligned}$$

If we bound the remaining Gaussian by 1 to get

$$|K_\theta(y,\tilde{y})| \le \lambda^{n+1-\Re\theta(2\rho+1)} (1+\lambda|y-\tilde{y}|)^{\Re\theta-\frac{n+1}{2}} \tag{3.27}$$

Now set $\Re\theta = \theta_1$ and observe that the following relations hold

$$\theta_1 - \frac{n+1}{2} = -\frac{2}{p_1}, \qquad n+1-\theta_1(2\rho+1) = \frac{2}{p_1}$$

This follows from the relations

$$\theta - \frac{n+1}{2} = -\frac{2}{p} - \frac{n-1}{q}, \qquad n+1-\theta(2\rho+1) = \frac{2}{p} + \frac{2n}{q}$$

which, by (3.16), hold for the first two points in (3.24), therefore they must also hold for the third.

Hence we obtain the kernel bound

$$|K_\theta(y,\tilde{y})| \le c|y-\tilde{y}|^{-\frac{2}{p_1}} \qquad \Re\theta = \theta_1 \tag{3.28}$$

which, by the Hardy-Littlewood-Sobolev inequality, gives the $L^{p_1'}(L^1) \to L^{p_1}(L^\infty)$ bound for $p_1 > 2$.

(iii) The endpoint estimate. To prove (3.22) in this case we adapt Keel and Tao's argument in [7] to our context. For later use it is useful to summarize the argument in the following

Theorem 4 *Let V be an integral operator satisfying the following two conditions:*

i) For al non-endpoint sharp Strichartz pairs (ρ, p, q) we have

$$\|V\|_{L^{p'}(L^{q'}) \to L^\infty(L^2)} \le c\lambda^{1+\rho} \tag{3.29}$$

and

$$\|V\|_{L^1(L^2) \to L^p(L^q)} \le c\lambda^{1+\rho} \tag{3.30}$$

(ii) The kernel $H(y, \tilde{y})$ of V satisfies

$$|H(y, \tilde{y})| \le \lambda^{n+1}(1 + \lambda|y - \tilde{y}|)^{-\frac{n-1}{2}} \tag{3.31}$$

Then for the end-point Strichartz pair $(\rho(r), 2, r)$ we have

$$\|V\|_{L^2(L^{r'}) \to L^2(L^r)} \le c\lambda^{1+2\rho(r)} \tag{3.32}$$

In our case we apply the theorem to $V = V_\lambda$. Then part (i) follows from Remark 3.1, while the kernel bound in (ii) follows from (3.27) for $\theta = 1$ since

$$H(y, \tilde{y}) = K_1(y, \tilde{y})\lambda^{1+2\rho}$$

Proof: First we decompose the operator V into

$$V = \sum V^j$$

whose kernels are supported in the region $|y - \tilde{y}| \approx 2^j$. To achieve this we partition $R^{n+1} \times R^{n+1}$ into dyadic cubes whose size is proportional to the distance to the diagonal.

More precisely, consider the set $\mathcal{Q}$ of all closed cubes which have size 2^j and vertices in $2^j Z^{2(n+1)}$ for some $j \in Z$, and which do not intersect the diagonal. $\mathcal{Q}$ is ordered by inclusion, and two cubes in $\mathcal{Q}$ are either (almost) disjoint or included one in another. Denote by $\mathcal{R}$ the maximal cubes in $\mathcal{Q}$. Then it is easy to see that, modulo sets of measure 0,

$$R^{n+1} \times R^{n+1} = \bigcup_{Q \in \mathcal{R}} Q$$

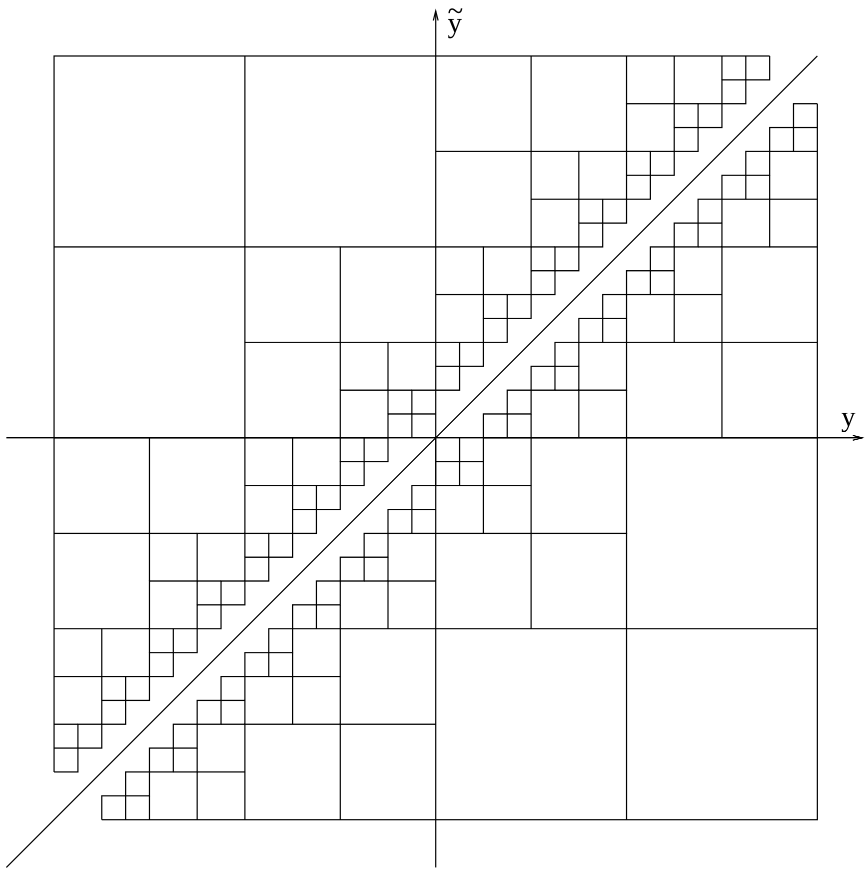

Figure 1: The dyadic decomposition for $R \times R$

is a disjoint partition of $R^{n+1} \times R^{n+1}$. In one dimension this decomposition is shown in Figure 1.

We label the cubes in R by their size,

$$R = \{Q_j^\alpha \times \tilde{Q}_j^\alpha \mid \; j \in Z, \;\; \alpha \in I\}$$

where for each α, j, the $n+1$ dimensional cubes Q_j^α, $\tilde{Q}_j^\alpha$ have size 2^j. Then we have the partition of unit into characteristic functions

$$1 = \sum_{j \in Z} \sum_\alpha \chi_{Q_j^\alpha} \chi_{\tilde{Q}_j^\alpha}$$

Correspondingly, set

$$V^j = \sum_\alpha \chi_{Q_j^\alpha} V_\lambda \chi_{\tilde{Q}_j^\alpha}$$

The kernel of V^j is supported at distance 2^j from the diagonal. In our case, the interesting scales are those for which $\lambda^{\frac{1}{2}} \leq 2^{-j} \leq \lambda$. For larger j the Gaussians take over and provide exponential decay, while for smaller j nothing happens since our operator acts roughly at frequency λ.

The key estimate, which is the analogue of Lemma 4.1 in [7], is:

Lemma 3.2 *The following estimate holds for $(q, \tilde{q})$ in a neighbourhood of $r = \frac{2(n-1)}{n-3}$:*

$$\|V^j\|_{L^2(L^{q'}) \to L^2(L^{\tilde{q}})} \leq \lambda^{1+\rho(q)+\rho(\tilde{q})} 2^{-j\beta(q,\tilde{q})} \tag{3.33}$$

Here

$$\beta(q, \tilde{q}) = \frac{n-3}{2} - \frac{n-1}{2}\left(\frac{1}{q} + \frac{1}{\tilde{q}}\right)$$

and

$$\rho(q) = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)$$

which corresponds to choosing ρ as in (3.16).

Since $\beta(r, r) = 0$, if we set directly $q = \tilde{q} = r$ in the Lemma then we get an uniform bound for the operators V_λ^j but we cannot sum up with respect to j. However, this summation can be fixed by using the same bilinear real interpolation argument as in [7]. It remains to prove the Lemma.

Proof of Lemma 3.2: Observe first that, by orthogonality with respect to cubes of size 2^j, it suffices to prove the estimate for a fixed cube,

$$\|\chi_{Q_j^\alpha} V \chi_{\tilde{Q}_j^\alpha}\|_{L^2(L^{q'}) \to L^2(L^{\tilde{q}})} \leq \lambda^{1+\rho(q)+\rho(\tilde{q})} 2^{-j\beta(q,\tilde{q})} \tag{3.34}$$

We show that this happens in two cases:

(a) If $q = \tilde{q} = \infty$

(b) If $\tilde{q} = 2,\ 2 \leq q < r$ and the dual range $q = 2,\ 2 \leq \tilde{q} < r$.

Then (3.34) follows by interpolation.

(a) If we use (3.31) then

$$\begin{aligned}
\|\chi_{Q_j^\alpha} V \chi_{\tilde{Q}_j^\alpha} v\|_{L^2(L^\infty)} &\leq 2^{\frac{j}{2}} \|\chi_{Q_j^\alpha} V \chi_{\tilde{Q}_j^\alpha} v\|_{L^\infty} \\
&\leq 2^{\frac{j}{2}} \|\chi_{Q_j^\alpha}(y) H(y, \tilde{y}) \chi_{\tilde{Q}_j^\alpha}(\tilde{y})\|_{L^\infty} \|\chi_{\tilde{Q}_j^\alpha} v\|_{L^1} \\
&\leq 2^{\frac{j}{2}} \lambda^{\frac{n+3}{2}} 2^{-j\frac{n-1}{2}} \|\chi_{\tilde{Q}_j^\alpha} v\|_{L^1} \\
&\leq 2^j \lambda^{\frac{n+3}{2}} 2^{-j\frac{n-1}{2}} \|v\|_{L^2(L^1)}
\end{aligned}$$

On the third line we use the bound on the kernel H, together with the fact that $|y-\tilde{y}| \approx 2^j$ for $y \in Q_j^\alpha$, $\tilde{y} \in \tilde{Q}_j^\alpha$. Then the conclusion (3.34) follows from the obvious equalities

$$\frac{n+3}{2} = 1 + \rho(\infty) + \rho(\infty) \qquad 1 - \frac{n-1}{2} = \beta(\infty,\infty)$$

(b) For $2 \leq q < r$ choose p as in (3.16) so that $(\rho(q), p, q)$ is a non-endpoint sharp Strichartz pair. Then use (3.29) to estimate

$$\begin{aligned} \|\chi_{Q_j^\alpha} V \chi_{\tilde{Q}_j^\alpha} v\|_{L^2} &\leq 2^{\frac{j}{2}} \|V \chi_{\tilde{Q}_j^\alpha} v\|_{L^\infty(L^2)} \\ &\leq 2^{\frac{j}{2}} \lambda^{1+\rho(2)+\rho(q)} \|\chi_{\tilde{Q}_j^\alpha} v\|_{L^p(L^q)} \\ &\leq 2^{j(1-\frac{1}{p})} \lambda^{1+\rho(2)+\rho(q)} \|v\|_{L^2(L^q)} \end{aligned}$$

On the first and the last line we have used the fact that the cubes Q_j^α, $\tilde{Q}_j^\alpha$ have size 2^j; however, in these estimates it does not matter that the two cubes are at distance 2^j. To conclude it remains to verify that

$$-1 + \frac{1}{p} = \beta(2,q)$$

which follows easily from the second part of (3.16). The second part of (b) follows in a similar manner from (3.30).

Observe that the above estimates are not the best ones to use for nonlinear equations since there we have additional L^p information about the regularity of the gradient of the coefficients. Following we present the one estimate based on this idea which is used later on for the study of quasilinear hyperbolic equations. The reader should then be easily able to produce other versions of it.

Theorem 5 *Assume that P is in divergence form and the coefficients satisfy $\nabla g \in L^2(L^\infty)$. Let (ρ, p, q) be a Strichartz pair. Then the following estimate holds locally:*

$$\|D^{1-\rho-\frac{\gamma}{2p}} u\|_{L^p(L^q)} \leq (1 + \|\nabla g\|_{L^2(L^\infty)}) \|\nabla u\|_{L^\infty(L^2)} + \|Pu\|_{H^{-\frac{\gamma}{p}}} \tag{3.35}$$

for

$$\begin{array}{ll} \gamma = 1 & n \geq 4 \\ \gamma > 1 & n = 3 \\ \gamma = 2 & n = 2 \end{array}$$

whenever the right hand side is finite.

Proof: For $(\rho, p, q) = (0, \infty, 2)$ this is the straightforward energy estimate. To get the remaining estimates we interpolate this with

$$\|D^{1-\rho-\frac{1}{4}}u\|_{L^p(L^q)} \leq (1 + \|\nabla g\|_{L^2(L^\infty)})\|\nabla u\|_{L^\infty(L^2)} + \|Pu\|_{H^{-\frac{1}{2}}} \tag{3.36}$$

The above value of γ follows from the choice of p above. To get the best result one would like to take p as small as possible. If $n = 2$ the best p is 4, if $n = 3$ then $p > 2$ and if $n > 3$ then we can take $p = 2$.

The proof of (3.36) is almost identical to the previous proof for $s = 1$ but uses instead Theorem 2 for the error estimates.

Remark 3.3 *Our argument fails for $n = 3$ and $\gamma = 1$ because we cannot use the forbidden endpoint $(0, 2, \infty)$ in the interpolation. However, it is conceivable that the estimate (3.35) is still true.*

4 Quasilinear hyperbolic equations

Consider a quasilinear second order hyperbolic equation in $R^n \times R$,

$$\partial_i g^{ij}(u)\partial_j u = N(u, \partial u) \tag{4.37}$$

with Cauchy data

$$u(0) = u_0, \qquad u_t(0) = u_1 \tag{4.38}$$

Then the classical theory (see [5], and also [20] and references therein) says that this problem is locally well-posed in $H^s \times H^{s-1}$ for $s > \frac{n}{2} + 1$. This condition insures that the coefficients of the principal part are C^1 and that ∇u is bounded.

Our goal is to use the new Strichartz estimates to obtain the same result for a lower s. Of course, we cannot do this for a general nonlinearity, for this would essentially require ∇u to be bounded. Hence, we shall confine ourselves to a quadratic nonlinearity of the form

$$N(u, u) = G(u)Q(\nabla u, \nabla u) \tag{4.39}$$

With this special type of nonlinearity the equation has some scaling, which corresponds to the function u being dimensionless, and to $s = \frac{n}{2}$. To avoid distracting technicalities we assume that the functions G, g^{ij} are smooth, bounded and have bounded derivatives up to a sufficiently high order. Also we assume that the coefficients g^{ij} are uniformly hyperbolic in time. Then our main result is

Theorem 6 *The quasilinear problem (4.37)-(4.38) is locally well-posed in $H^s \times H^{s-1}$ for*

$$s \geq \frac{n}{2} + \frac{7}{8} \quad , \quad n = 2$$
$$s > \frac{n}{2} + \frac{3}{4} \quad , \quad n \geq 3$$

This is the first result to go below the classical (L^2) theory. After obtaining these results we have learned that similar results were independently proved by H. Bahouri and J-Y. Chemin [1], using a different method.

It is useful to compare this with the similar results for the corresponding semilinear equation

$$\Box u = |\nabla u|^2 \tag{4.40}$$

for which the local theory is well-understood. This is summarized in the following table, which indicates the threshold above which local well-posedness holds. The numbers in brackets represent the results we conjecture to be true.

	The semilinear problem (4.40)		The quasilinear problem (4.37)	
n	using only Strichartz estimates	best result	using only Strichartz estimates	best result
2	$\frac{n}{2}+\frac{3}{4}$	$\frac{n}{2}+\frac{3}{4}$	$\frac{n}{2}+\frac{7}{8}$	$(\frac{n}{2}+\frac{7}{8})$
3	$\frac{n}{2}+\frac{1}{2}$	$\frac{n}{2}+\frac{1}{2}$[1]	$\frac{n}{2}+\frac{3}{4}$	$(\frac{n}{2}+\frac{3}{4})$
4	$\frac{n}{2}+\frac{1}{2}$	$\frac{n}{2}+\frac{1}{4}$[2]	$\frac{n}{2}+\frac{3}{4}$	$(\frac{n}{2}+\frac{5}{8})$
5+	$\frac{n}{2}+\frac{1}{2}$	$\frac{n}{2}$	$\frac{n}{2}+\frac{3}{4}$	$(\frac{n}{2}+\frac{1}{2})$

One should note, though, that at this point the counterexamples for the quasilinear equation are no better than those for the semilinear equation, see [9], [10].

Better results can be obtained in the semilinear case when the right hand side of the equation contains special quadratic forms $Q(\nabla u, \nabla u)$ which exhibit certain cancellation properties and are called null forms. A similar improvement seems reasonable in the quasilinear case.

[1] This can be easily proved in the framework of the $X^{s,\theta}$ spaces

[2] see [18]

However, the definition of the "null condition" in this context is more delicate than in the semilinear case, and beyond the purpose of this article.

The same method can be used for second order hyperbolic equations of the form

$$g^{ij}(u, \nabla u)\partial_i\partial_j u = N(u, \nabla u) \tag{4.41}$$

Differentiating once we obtain equations which are essentially of the form (4.37), therefore

Theorem 7 *The quasilinear problem (4.41)-(4.38) is locally well-posed in $H^s \times H^{s-1}$ for*

$$\begin{aligned} s &\geq \frac{n+2}{2} + \frac{7}{8} \quad , \quad n = 2 \\ s &> \frac{n+2}{2} + \frac{3}{4} \quad , \quad n \geq 3 \end{aligned}$$

Proof of Theorem 6:

A simple but useful first step is to reduce the problem to the case when the initial data is small. This can be achieved by rescaling; the equation is invariant with respect to the rescaling

$$u(t, x) \to u(\epsilon t, \epsilon x)$$

Thus the problem is reduced to the case when the initial data is small but only in the homogeneous space $\dot{H}^s \times \dot{H}^{s-1}$. Next take advantage of the finite speed of propagation to localize the initial data to a bounded set, say $B = B(0, 1)$. Then we need to argue that the initial data is small in $H^s(B) \times H^{s-1}(B)$. Since we already know that the homogeneous norms are small, it suffices to verify that the $L^2 \times L^2$ norms of the initial data are small. Assume without any restriction in generality that $\frac{n}{2} < s < \frac{n}{2} + 1$. Due to the Sobolev embeddings, $\dot{H}^{s-1} \subset L^p$ for some $p > 2$ and the L^2 smallness for u_1 in B follows. This argument has to be modified for $u_0 \in \dot{H}^s$ as the corresponding embedding $\dot{H}^s \subset C^\gamma$ no longer guarantees L^2 smallness in B. To achieve this we first need to subtract the constants, i.e. replace u_0 by, say, $u_0 - u_0(0)$ which is small in B. To conclude the reduction, it remains to observe that subtracting constants which are in a bounded set does not significantly change the problem.

Since we already know that local well-posedness holds for more regular data, our strategy is as follows:

(i) Prove a-priori H^s bounds for classical solutions.

(ii) Prove uniformly continuous L^2 dependence of the solutions on H^s initial data.

(iii) For any H^s initial data produce a local solution as an L^2 limit of classical solutions. Then the continuous dependence result extends to H^s solutions and gives uniqueness.

(i) **Energy estimates.** Define the elliptic operator $\langle D\rangle$ with symbol $(1+|\xi|^2)^{\frac{1}{2}}$. The crucial ingredient in the energy estimates is the commutator estimate

$$\|[P(x,u,D),\langle D\rangle^{s-1}]u\|_{L^2_x} \leq c\|\nabla u\|_{L^\infty}\|\nabla u\|_{H^{s-1}}, \qquad s \geq 1 \tag{4.42}$$

Indeed, we have

$$\begin{aligned} [P(x,D,u),\langle D\rangle^{s-1}]u &= [\langle D\rangle^{s-1}, g]D_x\nabla u \\ &= [D_x\langle D\rangle^{s-1}, g]\nabla u - \langle D\rangle^{s-1}(D_x g)\nabla u \end{aligned}$$

(To avoid dealing with two time derivatives in the commutator we can rewrite the equation so that $g^{00}=-1$). The Kato-Ponce estimate in [6] implies that

$$\|[D_x\langle D\rangle^{s-1}, g]\nabla u\|_{L^2} \leq c\|g\|_{Lip}\|\nabla u\|_{H^s} + \|g\|_{H^s}\|\nabla u\|_{L^\infty}$$

with

$$\|g(u)\|_{H^s} \leq c\|u\|_{H^s}$$

from the Moser estimates (In general the constant c would depend on $|u|_{L^\infty}$, but here we assume global bounds on g and sufficiently many of its derivatives). Hence (4.42) follows.

Given a function v we define

$$E(v) = \frac{1}{2}\int -g^{00}(u)|v_t|^2 + \sum_{i,j=1}^{n} g^{ij}v_{x_i}v_{x_j}dx$$

Then

$$\frac{d}{dt}E(v) =< P(x,u,D)v, v_t > + \int h(x,u)u_t|\nabla v|^2 dx$$

Apply this to $v = \langle D\rangle^{s-1}u$. Since

$$P(x,u,D)v = [\langle D\rangle^{s-1}, P(x,u,D)]u + \langle D\rangle^{s-1}G(u)(Du)^2,$$

we get

$$\frac{d}{dt}E(v) \leq c(\|v_t\|_{L^2}(\|[\langle D\rangle^{s-1}, P]u\|_{L^2} + \|G(u)(\nabla u)^2\|_{H^s}) + \|u_t\|_{L^\infty}E(v))$$

Now use (4.42) for the first term and the Moser estimates for the second term to obtain the classical energy estimates for the quasilinear wave equation,

$$\frac{d}{dt}E(v) \leq c\|Du\|_{L^\infty}E(v)$$

Hence,

$$\|(\nabla u)(T)\|_{H^s} \leq \|(\nabla u)(0)\|_{H^s} e^{c\int_0^T \|\nabla u(t)\|_\infty dt} \tag{4.43}$$

which shows that for $s > \frac{n}{2}+1$ the solution u can be continued in $H^{s+1}\times H^s$ as long as

$$\int_0^T \|\nabla u(t)\|_{L^\infty} < \infty$$

The energy estimate (4.43) was first proved by Klainerman [8] for integer s. Later the Kato-Ponce estimates in [6] allowed the extension of this result for non-integer s, see [20] and references therein.

$L^2(L^\infty)$ **estimates for** Du. Now we prove that Du stays in $L^2(L^\infty)$ for a time that depends only on the H^s norm of the initial data, $s > \frac{n}{2}+\frac{3}{4}$.

Lemma 4.1 *There exist $\epsilon, T_0 > 0$ so that any solution u in $[0,T]$ to (4.37) with smooth initial data so that*

$$\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} \leq \epsilon$$

satisfies

$$\|\nabla u\|_{L^2(0,T;L^\infty)} \leq 1$$

if $T \leq T_0$.

Given the energy estimates in (4.43), this shows in effect that the solution exists and is smooth for a time at least equal to T_0.

Proof: The Strichartz estimates in Theorem 5 applied to $\langle D\rangle^{s-2}\nabla u$ give

$$\|\langle D\rangle^{s-\theta-2}\nabla u\|_{L^2(L^p)} \leq c(1+\|\nabla g\|_{L^2(L^\infty)})\|\langle D\rangle^{s-2}\nabla^2 u\|_{L^\infty(L^2)} + \|P(x,u,D)\langle D\rangle^{s-2}\nabla u\|_{H^{-\frac{1}{2}}},$$

for some p, θ satisfying

$$\frac{1}{2} - \frac{1}{p} = \frac{\theta + \frac{5}{4}}{n}$$

which by the Sobolev embeddings and the commutator estimates (4.42) yield

$$\|\nabla u\|_{L^2(L^\infty)} \leq c(1+\|\nabla u\|_{L^2(L^\infty)})\|\nabla u\|_{L^\infty(H^{s-1})}$$

Note that in dimension $n = 3$ this should be replaced by an $L^{2+\epsilon}(L^p)$ estimate, while in dimension $n = 2$ this should be an $L^4(L^\infty)$ estimate. Using the energy estimate (4.43) we get

$$\|\nabla u\|_{L^2(L^\infty)} \leq c(1 + \|\nabla u\|_{L^2(L^\infty)})\|\nabla u(0)\|_{L^\infty(H^{s-1})} e^{c\sqrt{T}\|\nabla u\|_{L^2(L^\infty)}}$$

Hence if we choose ϵ and T_0 sufficiently small then we get

$$8\|\nabla u\|_{L^2(L^\infty)} \leq (1 + \|\nabla u\|_{L^2(L^\infty)}) e^{\|\nabla u\|_{L^2(L^\infty)}}$$

But this is false if $\|\nabla u\|_{L^2(0,T;L^\infty)} = 1$. Since this norm depends continuously on T and is 0 when $T = 0$, it follows that

$$\|\nabla u\|_{L^2(0,T;L^\infty)} < 1, \qquad T \leq T_0$$

q.e.d.

This concludes the proof of the a-priori estimates. As a consequence of the a-priori estimates, we get a lower bound on the lifespan of classical solutions which depends only on the $H^s \times H^{s-1}$ norm of the initial data.

(ii) The next step is to establish some continuous dependence on the data. More precisely, we shall prove the following

Lemma 4.2 *Let u, v be smooth solutions to (4.37) in $[0, T]$. Then*

$$\|\nabla(u - v)\|_{L^\infty(0,T;L^2)} \leq c(\|u|_{H^s}, \|v\|_{H^s}, \|\nabla u\|_{L^2(L^\infty)}, \|\nabla v\|_{L^2(L^\infty)})\|\nabla(u - v)(0)\|_{L^2} \tag{4.44}$$

Proof: The function $w = u - v$ solves the linear equation

$$P(x, u, D)w = A_0 w + A_1 \nabla w \tag{4.45}$$

where

$$A_0 = g(u, v)(D_x \nabla v) + G(u, v)(\nabla u, \nabla v)^2, \quad A_1 = G(u, v)(\nabla u, \nabla v)$$

The conclusion follows if we prove that (4.45) is well-posed in $H^1 \times L^2$, with a right hand side which is in L^2.

On one hand we estimate the coefficients A_0, A_1. For A_1 we easily get $A_1 \in L^2(L^\infty)$. For A_2 we use interpolation to bound the $D_x \nabla v$ term. We know that $\nabla v \in L^2(L^\infty)$ and $\langle D\rangle^{s-1}\nabla v \in L^\infty(L^2)$. Hence if we choose p_1, q_1 so that the following points are collinear:

$$\begin{array}{lll} (0 & \frac{1}{2} & 0) \\ (1 & \frac{1}{p_1} & \frac{1}{q_1}) \\ (s-1 & 0 & \frac{1}{2}) \end{array}$$

then we get $D_x \nabla v \in L^{p_1}(L^{q_1})$. The $(\nabla u, \nabla v)^2$ term is easier to estimate in the same space, therefore in the end we obtain $A_1 \in L^{p_1}(L^{q_1})$.

On the other hand, consider the equation

$$P(x,u,D)w = f \in L^2, \quad w(0) = w_0 \in H^1, \qquad w_t(0) = w_1 \in L^2$$

Due to the energy estimates we get $\nabla w \in L^\infty(L^2)$, while from the Strichartz estimates in Theorem 5 it follows that w satisfies

$$D^{2-s} w \in L^2(L^\infty)$$

Hence, if we choose (p_2, q_2) so that the following points are collinear

$$\begin{array}{lcc} (1 & 0 & \frac{1}{2}) \\ (0 & \frac{1}{p_2} & \frac{1}{q_2}) \\ (2-s & \frac{1}{2} & 0) \end{array}$$

then we get $u \in L^{p_2}(L^{q_2})$.

Matching the two pairs of collinear points it is easy to see that

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$$

Hence a simple fixed point argument shows that the equation (4.45) is L^2 well-posed and the right hand side is in L^2, q.e.d.

Remark 4.3 *For $n = 2$ the above argument has to be modified when $s < 2$, since then $D_x \nabla v$ can only be estimated in negative Sobolev spaces. Thus, the best result one can obtain in a similar manner is a stability estimate in $H^{\frac{7}{8}} \times H^{-\frac{1}{8}}$.*

(iii) Now we construct the solutions for data $(u_0, u_1) \in H^s \times H^{s-1}$. Let $(u_0^n, u_1^n) \in H^{N+1} \times H^N$ so that

$$(u_0^n, u_1^n) \to (u_0, u_1) \quad \text{in} \quad H^s \times H^{s-1}$$

Then the corresponding solutions u^n exist for some time T independent of n and ∇u_n are uniformly bounded in $L^\infty(H^{s-1})$ and $L^2(L^\infty)$.

Due to the stability estimate (4.44) the sequence u_n is L^2 convergent in $[0, T]$ to some function u. Then ∇u_n converges to ∇u weakly in $L^\infty(H^{s-1})$ and $L^2(L^\infty)$ but this suffices to insure that u solves the equation in the sense of distributions. Hence we can now use the energy estimates to conclude that the H^s norm of the solution, $\|\nabla u(t)\|_{H^s}$, is continuous. Since ∇u is weakly continuous in t, this insures that it is also strongly continuous in t, q.e.d.

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