OBERWOLFACH SEMINAR: DISPERSIVE EQUATIONS GEOMETRIC DISPERSIVE EVOLUTIONS

DANIEL TATARU

Contents

1. Introduction

Among the nonlinear dispersive equations, a distinguished class is that of geometric evolutions. Unlike the models seen earlier where nonlinear interactions are added to an underlying linear dispersive flow, here the nonlinear structure arises from the curvature of the state space itself. Precisely, our geometric evolutions are obtained by applying the standard linear Lagrangian or Hamiltonian formalism to a state space consiting of maps into (curved) manifolds.

The simplest geometric pde's are the elliptic and parabolic ones, namely the harmonic map equation and the harmonic heat flow. While these still play a role in our exposition, in these notes we are primarily concerned with the dispersive evolutions, the wave map equation and the Schrödinger map equation.

Both the short and the long time behavior of wave and Schrödinger maps are dependent on the curvature properties of the target manifold. Because of these, the model cases of maps into the sphere $\mathbb{S}^m$ and into the hyperbolic space play an important role.

Compared with other dispersive pde's, an additional structure present here is that of "gauge invariance". The simplest way this arises is in the choice of coordinates on the target manifold; also, in a more subtle way, in the choice of frames in the tangent space of the target manifold. Often a more favourable nonlinear structure is obtained by making a suitable gage choice. This is also related to the notion of "renormalization", which here represents a paradifferential version of choosing a good gauge.

The dimension of the underlying space-time affects the scaling and criticality properties of our equations. Our primary target here is the case of $2+1$ dimensions, which is arguably the most interesting. This is the energy critical case, i.e. for which the energy is invariant with respect to the natural scaling of the equations.

We begin these note with a brief description of the state space of maps into manifolds, followed by an introduction of the four main pde's, namely harmonic maps, the harmonic heat flow, wave maps and finally Schrödinger maps. Our main interest is in wave maps, where a series of developments in the last 15 years have led to a reasonably complete theory. We first discuss the small data case, where the emphasis is on function spaces and renormalization. Then we consider the large data problem, where in addition we bring in the concept of induction on energy, and study energy concentration using Morawetz estimates. Finally, the last section is concerned with the small data problem for Schrödinger maps, where the difficulties revolve around the gauge choice and function spaces. The large data problem for Schrödinger maps is still open.

2. Maps into manifolds

Instead of working with real or complex valued functions, our main objects of study here are evolutions whose state space, in the simplest setting, consists of maps from the Euclidean space $\mathbb{R}^n$ into a Riemannian manifold (M, g). More generally, one can consider maps whose domains are also Riemannian manifolds.

In terms of the target manifold (M, g), the most common situation we will consider is that of compact manifolds without boundary. Among these, the sphere $\mathbb{S}^2$ or its higher dimensional counterparts $\mathbb{S}^m$ will play the role of a model positively curved manifold. On such manifolds one often does not have a nice global coordinate chart. Thus, in order to describe global objects it is often convenient to view such manifolds, via Nash's theorem, as isometrically embedded into a higher dimensional Euclidean space,

$$(M, g) \hookrightarrow (\mathbb{R}^m, e)$$

We call this the **extrinsic setting**. The simplest such example is the unit sphere representation

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3;\ |x| = 1\} \subset \mathbb{R}^3$$

Among negatively curved manifolds, the model is the hyperbolic space $\mathbb{H}^2$ or more generally $\mathbb{H}^m$. While this is not compact, it can be viewed globally as embedded in the Minkowski space $(\mathbb{M}^{2+1}, m)$, with metric $ds^2 = -d\phi_0^2 + d\phi_1^2 + d\phi_2^2$

$$\mathbb{H}^2 = \{\phi \in \mathbb{M}^{2+1};\ |\phi|_m^2 = -1\} \subset \mathbb{M}^{2+1}$$

Alternatively, one can also use compact quotients of $\mathbb{H}^m$ as surrogates for $\mathbb{H}^m$. This is convenient if for instance one wants to adapt $\mathbb{H}^m$ to the extrinsic setting.

2.1. **The tangent bundle and covariant differentiation.** Given a differentiable map

$$\phi : \mathbb{R}^n \to (M, g)$$

we can define its partial derivatives with respect to the $\mathbb{R}^n$ coordinates at a point $x \in \mathbb{R}^n$, $\partial_i \phi(x) \in T_{\phi(x)}M$. These can be viewed as sections of a vector bundle E^ϕ over $\mathbb{R}^n$, where the fiber is given by $E^\phi_x = T_{\phi(x)}M$. Precisely, E^ϕ is a metric bundle, where the metric is inherited from TM.

On TM one has the Levi-Civita connection induced by the metric. Its pullback to $\mathbb{R}^n$ is a connection on E^ϕ. The easiest way to describe it is by using a local coordinate chart on M. If in a chart ϕ is given by $\phi = (\phi^1, \cdots \phi^m)$ and a section of E^ϕ is given by $v = (v^1, \cdots, v^m)$ then the covariant derivatives of v are given by

$$\mathbf{D}_j v^k(x) = \partial_j v^k(x) + \Gamma^k_{il} \partial_j \phi^i v^l(x) \tag{2.1}$$

Here Γ^k_{il} represent the Riemann-Christoffel symbols on M. This is a metric connection, i.e. $\mathbf{D}g = 0$. Another way to express this property is via the relation

$$\mathbf{D}_j \langle v, w\rangle_g = \langle \mathbf{D}_j v, w\rangle_g + \langle v, \mathbf{D}_j w\rangle_g$$

In particular one can consider the covariant derivatives of $\partial_j \phi$; there it is easy to establish that

D-commute

$$\mathbf{D}_i \partial_j \phi = \mathbf{D}_j \partial_i \phi \tag{2.2}$$

Of course the covariant derivatives themselves do not commute; instead the curvature $\mathbf{R}$ of the connection $\mathbf{D}$ is related to the curvature tensor R of M. Precisely, for any two sections v, w of E^ϕ we have the relation

-curvature

$$\langle [D_i, D_j] v, w\rangle_g = R(\partial_i \phi, \partial_j \phi, v, w) \tag{2.3}$$

Another way to express the covariant differentiation is in the context of the extrinsic setting. For this we assume that (M, g) is a submanifold of the Euclidean space $\mathbb{R}^m$. Then one can define the normal bundle NM. The second fundamental form $\mathcal{S}$ is a symmetric quadratic form

$$\mathcal{S} : TM \times TM \to NM$$

given by

$$\langle \mathcal{S}(X, Y), \nu\rangle = \langle \nabla_X Y, \nu\rangle = -\langle X\nu, Y\rangle$$

Here $X\nu$ is the X derivative of ν since the Euclidean space is flat. In this context, the connection $\mathbf{D}$ can be expressed in terms of the second fundamental form $\mathcal{S}$ as

$$\mathbf{D}_j v^k(x) = \partial_j v^k(x) + \mathcal{S}^k_{il} \partial_j \phi^i v^l(x) \tag{2.4}$$

By the Gauss-Codazzi equations, the curvature of the connection takes the form

$$\langle [D_i, D_j] v, w\rangle_g = \langle \partial_i \phi, v\rangle_g \langle \partial_j \phi, w\rangle_g - \langle \partial_j \phi, v\rangle_g \langle \partial_i \phi, w\rangle_g \tag{2.5}$$

2.2. **Special targets.** For the most part, the work so far in geometric dispersive equations is devoted to special targets, namely the sphere $\mathbb{S}^2$ (or $\mathbb{S}^m$) and the hyperbolic space $\mathbb{H}^2$ (or $\mathbb{H}^m$). The advantage is that the algebra is simpler, while one hopes that nothing fundamental is lost in the process. In both cases the preferred setting is the extrinsic one.

Consider first the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, and a map $\phi : \mathbb{R}^n \to \mathbb{S}^2$. By a slight abuse of notation we also use ϕ for the coordinates in $\mathbb{R}^3$. Then ϕ represents the unit outer normal to the sphere. The second fundamental form of the sphere is

$$\mathcal{S}(u, v) = -\langle u, v\rangle, \qquad u, v \perp \phi$$

The sections of E are $\mathbb{R}^3$ valued vector fields u with the property that $\langle u, \phi\rangle = 0$. The covariant derivatives are given by

$$D_j u = \partial_j u - \langle u, \partial_j \phi\rangle \phi \tag{2.6}$$

and their commutator is

$$[D_i, D_j]u = \langle \partial_i \phi, u\rangle \partial_j \phi - \langle \partial_j \phi, u\rangle \partial_i \phi$$

The case of $\mathbb{H}^2$ is almost identical. Representing it as the space-like hyperboloid

$$-\phi_0^2 + \phi_1^2 + \phi_2^2 = -1$$

in the Minkowski space $(\mathbb{M}^{2+1}, m)$, the upward normal is still given by ϕ and the above formulas for covariant differentiation remain unchanged provided that the inner products are now taken with respect to the Minkowski metric.

2.3. **Sobolev spaces.** The question of characterizing the Sobolev regularity of maps between manifolds is not fully understood at this time, and many open problems exist. The discussion below is confined to the specific setting that is needed later in these notes. For further references we refer the reader to the survey paper MR2376670 [27].

The issue at hand is primarily to understand the H^s regularity of maps $\phi : \mathbb{R}^n \to (M, g)$. There is a natural scaling law associated with such maps,

$$\phi(x) \to \phi(\lambda x)$$

In terms of L^2 based Sobolev norms, the one with exactly this scaling law is the $\dot H^{\frac{n}{2}}$ norm. The problems which we will discuss later all have $\dot H^{\frac{n}{2}}$ as a critical (scale invariant) Sobolev norm. Hence most of our discussion will revolve around $\dot H^{\frac{n}{2}}$. We also care about higher regularity; to study that we will consider the spaces $\dot H^s \cap \dot H^{\frac{n}{2}}$ for $s > \frac{n}{2}$. Finally, in various contexts we need to measure the regularity of sections of the vector bundle E^ϕ. For this we will still use homogeneous Sobolev spaces $\dot H^s$ but here we will allow a range of s below $\frac{n}{2}$.

A key feature of the space $\dot H^{\frac{n}{2}}$ is that it is a threshold in terms of Sobolev embeddings. Precisely, the embedding $\dot H^{\frac{n}{2}} \subset L^\infty$ barely fails but instead we have $\dot H^{\frac{n}{2}} \subset VMO$, the space of functions with vanishing mean oscillation. So while $\dot H^{\frac{n}{2}}$ functions are not continuous, they are almost localized in the sense that on small sets they vary very little in average.

As it turns out, VMO is a borderline space as far as the topological properties of maps are concerned. Precisely, the homotopy of VMO maps is well defined, and one can use the homotopy classes in order to partition VMO (and also $\dot H^{\frac{n}{2}}$) into connected components.

Another consequence of working with $\dot H^{\frac{n}{2}}$ is that it is not possible to confine the range of a map to the domain of a local chart on M, not even locally. Thus the extrinsic setting seems far more desirable from this perspective.

The space of maps $\phi : \mathbb{R}^n \to (M, g)$ is not a linear space, so one cannot endow it with a norm. There are two main methods to define the class of $\dot H^{\frac{n}{2}}$ maps:

In the extrinsic setting where we have a uniform isometric embedding $(M, g) \hookrightarrow (\mathbb{R}^m, e)$. There one can simply view maps $\phi : \mathbb{R}^n \to M$ as maps $\phi : \mathbb{R}^n \to \mathbb{R}^m$ which happen to take values in M. Then their regularity, as well as the regularity of sections of E^ϕ, is computed on components as real valued functions.

This is the most convenient setting to use in the analysis. The disadvantage is that it is not at all obvious whether this definition is geometric or whether it depends on the embedding at hand.

In the geometric setting. The easier case is when n is even. Then for ϕ smooth and constant outside a compact set one can define the homogeneous $\dot H^k$ Sobolev size of ϕ by

$$\|\phi\|^2_{\dot H^k} = \sum_j \sum_{|\alpha|=k-1} \|\mathbf{D}^\alpha \partial_j \phi\|^2_{L^2}, \qquad k \geq \frac{n}{2}$$

Then one can define the set of $\dot H^{\frac{n}{2}}$ maps by taking, say, L^2_{loc} limits of sequences which have bounded size in the above sense.

One can also endow the vector bundle E with a related norm. Precisely, for $v \in E$ we set

$$\|v\|^2_{\dot H^k} = \sum_{|\alpha|=k} \|\mathbf{D}^\alpha v\|^2_{L^2}, \qquad 0 \leq k \leq \frac{n}{2}$$

In the case of odd n one needs to work with fractional spaces, and for that it is necessary to consider a more roundabout route. This is based on the Littlewood-Paley theory. To describe the idea we begin with a complex valued function $\phi : \mathbb{R}^n \to \mathbb{C}$. To ϕ we associate its Littlewood-Paley truncations $\phi_{\leq k}$ to frequencies less than 2^k, as well as its dyadic pieces $\phi_k = \frac{d}{dk}\phi_{<k}$, where k is a real dyadic frequency parameter. Then for any large N we have

$$\|\phi\|^2_{\dot H^s} = c_{s,k} \int_{-\infty}^{\infty} 2^{2sk}\|\phi_k\|^2_{L^2} + 2^{2(s-N)k}\|\phi_k\|^2_{\dot H^N}\, dk$$

If, instead of taking $\phi_{<k}$ to be the exact Littlewood-Paley localization of ϕ, one takes an arbitrary smooth function which decays to 0 as $k \to -\infty$ and converges to ϕ as $k \to \infty$, then the above equality becomes an inequality,

$$\|\phi\|^2_{\dot H^s} \lesssim \int_{-\infty}^{\infty} 2^{2sk}\|\phi_k\|^2_{L^2} + 2^{2(s-N)k}\|\phi_k\|^2_{\dot H^N}\, dk$$

Then the $\dot H^s$ norm of ϕ can be defined by minimizing the right hand side with respect to all extensions $\phi_{<k}$ of ϕ as above,

$$\|\phi\|^2_{\dot H^s} \approx \inf_{\phi_{<k}} \int_{-\infty}^{\infty} 2^{2sk}\|\frac{d}{dk}\phi_{<k}\|^2_{L^2} + 2^{2(s-N)k}\|\frac{d}{dk}\phi_{<k}\|^2_{\dot H^N}\, dk$$

The above definition involves only integer H^s norms, and it carries over easily to our context. Precisely, given a measurable map

$$\phi_0 : \mathbb{R}^n \to M$$

we call a smooth function

$$\phi : \mathbb{R} \times \mathbb{R}^n \to M$$

an admissible extension of ϕ_0 if $\lim_{k\to\infty}\phi(k) = \phi_0$ in L^2, and $\lim_{k\to-\infty}\nabla\phi(k) = 0$. Then we set

$$\|\phi_0\|_{\dot H^s} = \inf_{\phi \ admissible} \int_{-\infty}^{\infty} 2^{2sk}\|\partial_k\phi(k)\|_{L^2}^2 + 2^{2(s-N)k}\|\partial_k\phi(k)\|_{\dot H^N}^2 \, dk$$

A similar definition applies to sections of E^{ϕ_0}. There one needs to consider also extensions to sections of E^{ϕ}.

An alternate route is to consider a distinguished extension rather that all possible extensions. A suitable one is given for instance by the harmonic heat flow described below.

To compare the above $\dot H^s$ classes of maps we have the following:

Theorem 2.1 ([48]). *The extrinsic $\dot H^{\frac{n}{2}}$ class and the geometric $\dot H^{\frac{n}{2}}$ class are equivalent for small $\dot H^{\frac{n}{2}}$ sizes. In the same context, the higher regularity classes of maps $\dot H^s \cap \dot H^{\frac{n}{2}}$ are also equivalent.*

Likely this correspondence extends to all maps in the zero homotopy class. Unfortunately the geometric definition, as stated, applies only to homotopy zero maps.

2.4. $\mathbb{S}^2$ **and targets: homotopy classes and equivariance.** As mentioned before, the family $\dot H^{\frac{n}{2}}$ maps is divided into connected components, indexed by the homotopy class. One model case of interest is that of maps $\phi : \mathbb{R}^2 \to \mathbb{S}^2$. There the homotopy class is indexed by integers m, computed via the formula

$$\int_{\mathbb{R}^2} \phi\cdot(\partial_1\phi\times\partial_2\phi)dx = 4\pi m$$

Here the integrand is exactly the pull-back of the volume form on $\mathbb{S}^2$, and the integral is finite for all finite energy maps by the Cauchy-Schwarz inequality. Intuitively this measures the number of times the map ϕ wraps around the sphere.

We remark that in the case of the $\mathbb{H}^2$ target all the finite maps have homotopy zero, and the direct analogue of the above integral vanishes.

In many difficult nonlinear pde's one can gain insights by studying classes of solutions which have additional symmetries. Often one uses the class of radial solutions. In our case, spherically symmetric maps are less useful, in part because they have homotopy zero (the integrand above is in fact identically zero). Instead, the interesting class of maps is the **equivariant** class.

The **equivariant** maps are maps which, when expressed in polar coordinates, satisfy

$$\phi(r,\theta) = (u(r), k\theta + \theta_0(r)), \qquad u\in[0,\pi] \tag{2.7}$$

where k is the equivariance class. Another interpretation of this is the relation

$$\phi(Rx) = R^k\phi(x)$$

where R stands at the same time for a rotation around the origin in $\mathbb{R}^2$, respectively a rotation around the N-S axis in $\mathbb{S}^2$.

Here $k = 0$ corresponds to radial symmetry. If $k \neq 0$ then all $\dot H^{\frac{n}{2}}$ equivariant maps must have a limit at 0 and at infinity, which can be either pole, S or N. The homotopy index is then a multiple of the equivariance class.

We also remark that, in a more restrictive interpretation, sometimes one defines equivariant maps as maps of the form

$$\phi(r,\theta) = (u(r), k\theta + \theta_0) \tag{2.8}$$

This works for harmonic maps, the harmonic heat flow and for wave maps. However this restricted class is not invariant with respect to the Schrödinger map flow.

2.5. **Frames and gauge freedom.** This approach to the study of maps from $\mathbb{R}^n$ into manifolds begins with a choice of an orthonormal frame $\{e_k(\phi)\}$ in $T_\phi M$. Then the idea is to describe the map ϕ via its gradient expressed in this frame. We obtain the **differentiated fields** ϕ_α given by

$$\psi_{\alpha,k} = \langle \partial_\alpha \phi, e_k \rangle_g$$

To start with, these satisfy the compatibility conditions

$$\mathbf{D}_\alpha \psi_\beta = \mathbf{D}_\beta \psi_\alpha, \tag{2.9}$$

where the new covariant differentiation operators $\mathbf{D}_\alpha$ expressed in the frame have the form

$$\mathbf{D}_\alpha = \partial_\alpha + A_\alpha.$$

Here the **connection coefficients** A_α are antisymmetric matrices given by

$$(A_\alpha)_{jk} = \langle e_j, D_\alpha e_k \rangle_g$$

A-priori the coefficients A_α satisfy the curl system

$$(\partial_\alpha A_\beta - \partial_\beta A_\alpha)_{jk} = R(\partial_\alpha \phi, \partial_\beta \phi, e_j, e_k) = \psi_{\alpha,i}\psi_{\beta,l} R(e_i, e_l, e_j, e_k) \tag{2.10}$$

where R is the Riemann curvature tensor on (M, g). This is not yet a well determined system because the orthonormal frame has not been specified. Varying the frame choice leads to the gauge invariance

$$\psi_\alpha \to \mathcal{O}\psi_\alpha, \qquad A_\alpha \to \mathcal{O} A_\alpha \mathcal{O}^{-1} - \partial_\alpha \mathcal{O}\mathcal{O}^{-1}, \qquad \mathcal{O} \in SO(m)$$

Specifying an orthonormal frame is called fixing the gauge.

Assuming that M is parallelizable, one natural option would be to consider a fixed frame which is tied to M. However, this does not improve at all the analysis, and defeats the purpose of trying to express all equations exclusively in terms of the differentiated fields ψ_α. Indeed, the main advantage of the frame method is that one can produce equations with a better structure by choosing a favorable frame which depends not only on M but also on the map ϕ.

Another obstruction to the above goal has to do with the fact that in general curvature tensor in (2.10) depends on the original map ϕ. However, there is one interesting case when we do obtain a self-contained system, namely when M has constant curvature κ. Then the system (2.10) can be rewritten in the simpler form

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha = \kappa(\psi_\alpha \otimes \psi_\beta - \psi_\beta \otimes \psi_\alpha) \tag{2.11}$$

For this reason, the frame method has been primarily used so far in the case when M is either the sphere or the hyperbolic space.

An obvious way to complete this system and uniquely determine A is to add the divergence relation

$$\partial_\alpha A_\alpha = 0 \tag{2.12}$$

This is called the **Coulomb gauge**. Then A_α are uniquely determined by (2.10) and (2.12), namely

$$A_\alpha = -\frac{1}{2}\kappa \Delta^{-1} \partial_\beta (\psi_\alpha \otimes \psi_\beta - \psi_\beta \otimes \psi_\alpha) \tag{2.13}$$

A further simplification occurs when the target manifold is two dimensional. Then $\psi_\alpha \in \mathbb{R}^2$, which we identify with $\mathbb{C}$. On the other hand A_α can be viewed as real rotation coefficients. Then the ψ_α belong to a complex vector bundle over $\mathbb{R}^n$ endowed with the connection

$$D_\alpha = \partial_\alpha + iA_\alpha$$

The curl relations (2.10) become

curlaccb

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha = \kappa \Im(\psi_\alpha \bar{\psi}_\beta) \tag{2.14}$$

and the gauge freedom translates to

$$\psi_\alpha \to e^{i\chi}\psi_\alpha, \qquad A_\alpha \to A_\alpha + \partial_\alpha \chi$$

where χ is any real valued function. In the Coulomb gauge the connection coefficients are given by

coulombb

$$A_\alpha = -\frac{1}{2}\kappa \Delta^{-1}\partial_\beta \Im(\psi_\alpha \bar{\psi}_\beta) \tag{2.15}$$

As a final remark here, the Coulomb gauge works well in high dimension (say $n \geq 4$). However, in low dimensions there are issues associated to *high* $\times$ *high* $\to$ *low* frequency interactions in the above expression for A, and new gauge choices are needed. The situation improves somewhat if one considers maps with extra symmetries (e.g. equivariant).

3. Geometric pde's

3.1. **Harmonic maps.** We first review the linear Laplace equation. For functions

$$\phi : \mathbb{R}^n \to \mathbb{R}$$

we define the Lagrangian

el:lag

$$L^e(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla_x \phi|^2 \, dx = \frac{1}{2}\int_{\mathbb{R}^n} \partial_i \phi \cdot \partial_i \phi \, dx, \tag{3.1}$$

with the Euclidean summation convention. Local critical points solve the corresponding Euler-Lagrange equation, which is the Laplace equation.

$$-\Delta \phi = 0 \qquad \text{or} \qquad -\partial_j \partial_j \phi = 0$$

We now repeat the above process, but with the key difference that instead of considering maps ϕ which take real or complex values, we consider maps which take values into a Riemannian manifold (M, g). The analogue of the elliptic Lagrangian in (3.1) is

hm:lag

$$L^e(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} \langle \partial^\alpha \phi, \partial_\alpha \phi \rangle_g \;\; dx, \tag{3.2}$$

The associated Euler-Lagrange equation is called the **harmonic map** equation, and is similar to the Laplace equation, namely

hm

$$-\mathbf{D}_j \partial_j \phi = 0 \tag{3.3}$$

where $\mathbf{D}_j$ are the covariant differentiation operators introduced in the previous section. Thus the above equation is no longer a linear equation; instead, as we shall see in a moment, it becomes a semilinear elliptic equation.

Expressed in local coordinates on the target manifold, the above equation takes the form

$$-\Delta\phi^i = \Gamma^i_{jk}(\phi)\partial_\alpha\phi^j\partial_\alpha\phi^k$$

This problem is invariant with respect to the dimensionless scaling

$$\phi(x) \to \phi(\lambda x)$$

therefore a natural translation invariant setting to study this problem is that of the Sobolev space $\dot{H}^{\frac{n}{2}}$. On the other hand the Lagrangian is invariant with respect to this scaling only if $n = 2$. We call that the **energy critical** problem. The higher dimensional case $n > 2$ is energy supercritical.

As mentioned before, another fact to consider is that $\dot{H}^{\frac{n}{2}}$ functions are not necessarily bounded. Hence there is no guarantee that any such map will stay locally within the domain of a local chart on M. This emphasizes the global aspects of the problem, and effectively eliminates the use of local coordinates in the study of the equation.

Switching to the extrinsic setting, the harmonic map equation takes the form

$$-\Delta\phi^i = \mathcal{S}^i_{jk}(\phi)\partial_\alpha\phi^j\partial_\alpha\phi^k$$

While just considering the above equation no additional structure is present, one has to also keep in mind the geometric properties of the second fundamental form. In particular we have the relation

$$\mathcal{S}^k_{ji}(\phi)\partial_\alpha\phi^k = 0$$

as one is a normal vector and the other is a tangent vector to M. Thus one can rewrite the equation in the form

$$-\Delta\phi^i = (\mathcal{S}^i_{jk}(\phi) - \mathcal{S}^k_{ji}(\phi))\partial_\alpha\phi^j\partial_\alpha\phi^k$$

which leads to the study of more general equations of the form

$$-\Delta\phi = \Omega_\alpha\partial_\alpha\phi$$

with the key property that $\Omega_\alpha \in \dot{H}^{\frac{n}{2}-1}$ are antisymmetric matrices.

From the perspective of geometric dispersive equations, harmonic maps are interesting as the steady states of the evolution problems. Thus it is useful to us to discuss the existence and regularity of harmonic maps. We begin with the local regularity question. In two dimensions this is provided by the following result for finite energy maps:

Theorem 3.1 (Hélein MR1913803 [19]). *Harmonic maps with locally finite energy are smooth in the energy critical case $n = 2$.*

The frame method and the Coulomb gauge have played a critical role in Heléin's approach. Their role is roughly to produce an elliptic equation with a perturbative nonlinearity. However, an alternate, more recent approach by Rivière MR2285745 [33] uses the extrinsic formulation of the problem. The higher dimensional counterpart of the above result is as follows[1] :

Theorem 3.2 (Evans MR1143435 [14], Bethuel MR1208652 [7]). *Local $\dot{H}^{\frac{n}{2}}$ harmonic maps are smooth.*

Secondly, we discuss the issue of existence of nontrivial finite energy harmonic maps in dimension $n = 2$. This is relevant since such maps are stationary solutions for wave and Schrödinger maps. The answer to this question depends on the geometry of the target

[1] Their results are actually stronger than stated here.

manifold. We consider two opposite examples. The first is the hyperbolic space $\mathbb{H}^2$, where we have the following Liouville type result:

Theorem 3.3 (Lemaire [24]). *There are no nontrivial finite energy harmonic maps from $\mathbb{R}^2$ into*[2] $\mathbb{H}^2$.

By contrast, the class of finite energy harmonic maps $\phi : \mathbb{R}^2 \to \mathbb{S}^2$ is quite rich. To describe it we first recall that the class of all finite energy maps $\phi : \mathbb{R}^2 \to \mathbb{S}^2$ consists of infinitely many connected components, indexed by their homotopy class $k \in \mathbb{Z}$ defined by

$$4\pi k = \int_{\mathbb{R}^2} \phi \cdot (\partial_1 \phi \times \partial_2 \phi) dx$$

This is finite since by Cauchy-Schwartz we have

$$4\pi |k| \lesssim \frac{1}{2} \int_{\mathbb{R}^2} |\partial_1 \phi|^2 + |\partial_2 \phi|^2 dx = E(\phi)$$

Within each homotopy class one can look for energy minimizers which turn out to have energy exactly $4\pi|k|$. In order for equality to hold above the two derivatives $\partial_1\phi$ and $\partial_2\phi$ are orthogonal and of equal size. This means that ϕ must be conformal. Such maps are nonunique due to the many symmetries of the problem. To remove some of the degrees of freedom we turn our attention to k- equivariant maps which take 0 to the south pole and infinity to the north pole. Then, for $k \neq 0$, one can find a k-equivariant harmonic map with energy $4\pi k$, namely

$$Q^k(r,\theta) = (2\tan^{-1}(r^k), k\theta), \qquad k \geq 1$$

which is unique modulo scaling and rotations.

3.2. **The harmonic heat flow.** Starting again with the Euclidean case, consider the gradient flow associated to the Lagrangian (3.1). We obtain the heat equation in $\mathbb{R} \times \mathbb{R}^n$, namely

$$(\partial_t - \Delta)\phi = 0 \qquad \text{or} \qquad (\partial_t - \partial_j \partial_j)\phi = 0, \qquad \phi(0) = \phi_0$$

The geometric analogue of this, namely the **harmonic heat flow**, is the gradient flow associated to the geometric Lagrangian (3.2). The equation has the form

$$\partial_t \phi - \mathbf{D}_j \partial_j \phi = 0, \qquad \phi(0) = \phi_0 : \mathbb{R}^n \to M \tag{3.4}$$

This is a semilinear parabolic equation for which L^e is a Lyapunov functional,

$$\frac{d}{dt} L^e(\phi) = -\int_{\mathbb{R}^n} \langle \mathbf{D}_i \partial_i \phi, \mathbf{D}_j \partial_j \phi \rangle_g dx$$

The associated scaling is

$$\phi(t,x) \to \phi(\lambda^2 t, \lambda x)$$

As before, this makes the problem energy critical in dimension $n = 2$, and supercritical in higher dimension.

In the extrinsic formulation the harmonic heat flow takes the form

$$(\partial_t - \Delta)\phi^i = \mathcal{S}^i_{jk}(\phi) \partial_\alpha \phi^j \partial_\alpha \phi^k, \qquad \phi(0) = \phi_0 \tag{3.5}$$

This is a semilinear parabolic equation with a nonlinear constraint, namely that $\phi(t,x) \in M$ for all $(x,t) \in \mathbb{R}^{n+1}$. Extending $\mathcal{S}$ in any fashion outside M one may also interpret this

[2] The same result holds for any negatively curved target

equation as a parabolic equation for $\mathbb{R}^m$ valued functions, where the above constraint is dynamically preserved.

We begin with the small data problem, for which one can directly use perturbative techniques to solve the equation:

Theorem 3.4 (Chen-Ding [9]). *Assume that the initial data u_0 for the harmonic heat flow is small in the critical Sobolev space $\dot H^{\frac{n}{2}}$. Then there is a unique global solution, which is smooth for $t > 0$.*

A similar result holds for data which is small in the larger space BMO, see [25].

Consider now the large data problem. In supercritical dimensions $n \geq 3$, blow up can occur in finite time in a self-similar manner. However, in the critical dimension $n = 2$ the self-similar blow-up is disallowed, and the only possibility for blow up is the "bubbling off" of harmonic maps, where a portion of the energy concentrates at a point close to a rescaled harmonic map, see Chen-Struwe [10] and Topping [49]. Precisely, we have the following result for energies below $E_{crit}(M)$, the lowest energy of a nontrivial harmonic map $\phi : \mathbb{R}^n \to M$:

Theorem 3.5 (Struwe [39], Qing-Tian [30], Smith [36]). *Let $n = 2$. Assume that the energy of the initial data u_0 for the harmonic heat flow is below $E_{crit}(M)$. Then there is a unique global solution, which is smooth for $t > 0$.*

In the particular case of the $\mathbb{H}^m$ target space, there are no nontrivial harmonic maps so there is a large data global well-posedness result. The case of the sphere $\mathbb{S}^2$ as a target is much richer. There we have at our disposal the equivariant harmonic maps Q_k described in the previous section, and a natural question is what happens for data which is close in energy to these. A result in [17] asserts that within the equivariant class the Q_k's are stable for $|k| \geq 3$. For $|k| = 2$ instability can occur, but there is no finite time blow-up [16]. Finally, one can have finite time blow-up for $k = 1$, see [31].

This seems to indicate that the generic blow-up pattern should be the bubbling off of single spheres, associated by a corresponding decrease in the homotopy class.

3.3. **Wave maps.** Formally, wave maps can be described by replacing the domain $\mathbb{R}^n$ used for harmonic maps by the the Minkowski space $\mathbb{M}^{n+1}$. For real valued functions $\mathbb{M}^{n+1}$ in the corresponding Lagrangian is

$$L^m(\phi) = \frac{1}{2}\int_{\mathbb{M}^{n+1}} -|\partial_t\phi|^2 + |\nabla\phi|^2 \; dxdt = \frac{1}{2}\int_{\mathbb{M}^{n+1}} \partial^\alpha\phi\partial_\alpha\phi \; dxdt, \tag{3.6}$$

where indices are lifted with respect to the Minkowski metric. The associated Euler-Lagrange equation is the wave equation in $\mathbb{M}^{n+1}$,

$$\Box\phi = 0, \qquad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1$$

where the d'Allembertian is given by

$$\Box = \partial_t^2 - \Delta_x = -\partial^\alpha\partial_\alpha.$$

For functions with values in a Riemannian manifold (M, g) we can consider a similar Lagrangian to the above one,

$$L^m(\phi) = \frac{1}{2}\int_{\mathbb{M}^{n+1}} \langle\partial^\alpha\phi, \partial_\alpha\phi\rangle_g \;\; dxdt,$$

The associated Euler-Lagrange equation is called the **wave map** equation, and has the form

wm
$$\mathbf{D}^\alpha \partial_\alpha \phi = 0, \qquad \phi(0) = \phi_0, \quad \partial_t \phi(0) = \phi_1. \tag{3.7}$$

This is a semilinear wave equation, for which the initial position and velocity are maps

$$\phi_0 : \mathbb{R}^n \to M, \qquad \phi_1 : \mathbb{R}^n \to T_{\phi_0} M$$

with $\phi_1 \in E^{\phi_0}$. The steady states of this evolution are precisely the harmonic maps discussed before.

A feature which is common with the linear wave equation is the conservation of the energy and momentum,

$$E(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} |\partial_x \phi|^2 + |\partial_t \phi|^2 \, dx, \qquad M_i(\phi) = \int_{\mathbb{R}^n} \partial_i \phi \cdot \partial_t \phi \, dx.$$

The scaling associated to this problem is

$$\phi(t,x) \to \phi(\lambda t, \lambda x)$$

so the scale invariant initial data space is $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$. Again, the most interesting case is the energy critical case, $n = 2$.

In addition, the wave map problem inherits the full Lorentz group of symmetries from the linear wave equation. Thus, in addition to steady states (harmonic maps), we also have their Lorentz transforms, which are waves with a fixed profile and constant velocity (less than 1). It is worth noting that taking a Lorentz transform of a harmonic map leads to an increase in energy.

In the extrinsic formulation the wave map equation is:

$$\Box \phi^i = -\mathcal{S}^i_{jk}(\phi) \partial^\alpha \phi^j \partial_\alpha \phi^k, \qquad \phi(0) = \phi_0, \quad \partial_t \phi(0) = \phi_1 \tag{3.8}$$

In the case of the $\mathbb{S}^m$ target this equation takes a very simple form,

$$\Box \phi = -\phi \, \langle \partial^\alpha \phi, \partial_\alpha \phi \rangle$$

A very similar formula holds for maps into $\mathbb{H}^m$,

$$\Box \phi = \phi \, \langle \partial^\alpha \phi, \partial_\alpha \phi \rangle_m$$

This problem is quite different from the corresponding heat flow, in that it is a **dispersive** equation. In other words, one has, on one hand, energy conservation, while, on the other hand linear waves travel (with speed one) in different directions and disperse. Hence, one does not expect, as in the parabolic case, a pure decay to a harmonic map pattern, but instead a more plausible picture is that of a splitting into one or more solitons (Lorentz transforms of harmonic maps) plus a dispersive part. While such a complete picture is not proved at the moment, considerable progress was made in recent years.

The first aim of the present notes is to describe the proof of the small data result:

small-data
Theorem 3.6 (Tao [45]: $\mathbb{S}^m$, Krieger [22]: $\mathbb{H}^2$, Tataru [48]: (M,g))**.** *The wave map equation is globally well-posed for initial data which is small in* $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$.

This is done in the next section. The result is briefly stated above. A more precise formulation requires the introduction of a suitable function space S for the solutions, associated to the initial data space $\dot H^{\frac{n}{2}} \times \dot H^{\frac{n}{2}-1}$. This is done later, but for now we mention the embedding

$$S \subset C(\mathbb{R}; \dot H^{\frac{n}{2}}) \cap \dot C^1(\mathbb{R}; \dot H^{\frac{n-1}{2}})$$

Expressed in terms of S, the above result includes:

- Existence: solutions exist in S.
- Uniqueness: solutions are unique in S.
- Continuous dependence: the map
$$(\dot H^{\frac{n}{2}} \times \dot H^{\frac{n}{2}-1}) \cap (\dot H^{\frac{n}{2}-\delta} \times \dot H^{\frac{n}{2}-1-\delta}) \ni (\phi_0, \phi_1) \to \phi \in S, \qquad \delta > 0$$
is continuous.
- Regularity: If in addition the data is in $\dot H^s \times \dot H^{s-1}$ for some $s > \frac{n}{2}$ then the solution stays uniformly bounded in the same norm.
- Scattering: after a suitable renormalization, the solutions approach a free wave at infinity.

The next question to ask is to what extent are the results in the small data case still valid for large data. One key difference in that regard occurs between the critical dimension $n = 2$ and supercritical dimensions $n \geq 3$. In two space dimensions the energy coincides with the critical Sobolev norm, and is a conserved quantity. In higher dimensions, on the other hand, there is no known mechanism to keep the critical Sobolev norm bounded; the energy is too weak for that purpose. Hence if $n \geq 3$ it makes sense to try to study solutions for which an uniform a-priori critical Sobolev bound is known.

An obstruction to having global scattering solutions comes from known solutions which either blow-up or do not decay as time goes to infinity. Such examples include:

- Self-similar solutions $\phi(t, x) = \phi(\frac{x}{t})$ blow up in finite time; many examples are known if $n \geq 3$, but such solutions cannot exist and have finite energy if $n = 2$.
- Solitons (harmonic maps and their Lorentz transforms) do not blow up, but cause scattering to fail.
- Soliton-like concentration; this can indeed occur even if $n = 2$, and is discussed in Section wm-further 4.9.

On the positive side, we do have the finite speed of propagation: if blow up occurs, it has to happen via critical Sobolev norm concentration at the tip of a light cone. This severely limits the possible blow-up geometries.

We begin our discussion with the two dimensional case, where the primary enemies for global solutions are the solitons, which correspond to harmonic maps. Then it is natural to introduce the following heuristic classification of target manifolds (M, g):

- No nonconstant harmonic maps $\Rightarrow$ **defocusing**, $E_{crit} = \infty$, e.g. $M = \mathbb{H}^m$.
- Nontrivial harmonic maps $\Rightarrow$ **focusing**, $E_{crit} < \infty$, e.g. $M = \mathbb{S}^m$.

In the defocusing case, one expects global well-posedness for large data. In the focusing case, global well-posedness should hold at least for data with energy below the ground state energy E_{crit}, i.e. the energy of the smallest nontrivial harmonic map. This has been known as the **Threshold Conjecture**, but is now a theorem:

Theorem 3.7 (Sterbenz-Tataru [37], [38]). *The following hold for the wave map equation in dimension $n = 2$:*

a) In the defocusing case we have global well-posedness and scattering for large data in $\dot{H}^1 \times L^2$.

b) In the focusing case we have global well-posedness and scattering for all data in $\dot{H}^1 \times L^2$ below the ground state energy E_{crit}.

The main ideas of the proof of this theorem are also presented in the next section. Prior to this, the same result was established in the equivariant case by Cote-Kenig-Merle [13]. Independently, the case $M = \mathbb{H}^m$ was treated by Tao, see [43] and further references therein, and the case $M = \mathbb{H}^2$ was treated by Krieger-Schlag [23].

3.4. **Schrödinger maps.** The Schrödinger equation is closely related to the heat equation, and can be obtained by allowing complex valued solutions for the heat equation and then extending those analytically in the half-space $\Re t \geq 0$. Restricting these solutions to the imaginary axis one obtains

$$(i\partial_t - \Delta)\phi = 0 \qquad \text{or} \qquad (i\partial_t - \partial_j\partial_j)\phi = 0, \qquad \phi(0) = \phi_0$$

The situation is slightly more complicated in the case of the Schrödinger maps. For that to make sense in the above context, we need a complex structure on the tangent space TM. Thus the natural setting is to have a Kahler manifold (M, g, J, ω) as a target. Even then, the Schrödinger map equation can no longer be obtained by taking a holomorphic extension of the harmonic heat flow in a half-space; indeed, the two flows no longer commute.

To introduce the Schrödinger map equation it is convenient to use the Hamiltonian formalism. In the case of the linear Schrödinger equation, the Hamiltonian is

$$H(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\phi|^2 \, dx$$

and the symplectic form is

$$\underline{\omega}(u, v) = \Im \int_{\mathbb{R}^n} u\bar{v} \, dx$$

For the Schrödinger map equation the Hamiltonian stays essentially unchanged

$$H(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\phi|_g^2 \, dx \tag{3.9}$$

while the symplectic form becomes

$$\underline{\omega}(u, v) = \int_{\mathbb{R}^n} \langle u, Jv\rangle_g \, dx = \int_{\mathbb{R}^n} \omega(u, v) \, dx, \qquad u, v \in E^\phi \tag{3.10}$$

The associated Hamilton flow is the **Schrödinger map** equation

$$\phi_t = JD^j\partial_j\phi, \qquad \phi(0) = \phi_0 \tag{3.11}$$

where J is the complex structure on TM.

The associated scaling law is the parabolic scaling,

$$\phi(t, x) \to \phi(\lambda^2 t, \lambda x)$$

and the scale invariant space for the initial data is again $\dot{H}^{\frac{n}{2}}$.

While the above form of the equation is fairly general, most of the work so far has been done for special targets, namely the sphere $\mathbb{S}^2$ and the hyperbolic space $\mathbb{H}^2$. In the case of the sphere the form of the equation is

$$\partial_t \phi = \phi \times \Delta \phi$$

where the cross product's purpose is twofold: to eliminate the component of $\Delta\phi$ which is normal to the sphere, and to rotate the remaining part by $\pi/2$. In the $\mathbb{H}^2$ case the equation looks identical except for a sign twist in the definition of the cross product.

The equation (3.11) admits one conserved quantity which is the counterpart of the usual energy functional for the linear Schrödinger equation:

$$E(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \phi|_g^2 \, dx$$

This is also the Hamiltonian; we use the terminology interchangeably.

In general there seems to be no direct counterpart of the conservation of mass and momentum; see however [15]. This can be related to the loss of the Galilean invariance.

The aim of the last section of these notes is to describe the proof of the small data result in critical Sobolev spaces:

Theorem 3.8 (Bejenaru-Ionescu-Kenig-Tataru [4])**.** *Consider the Schrödinger map equation with values into $\mathbb{S}^2$. Then global well-posedness holds for initial data which is small in the space $\dot H^{\frac{n}{2}}$.*

As for wave maps, this result includes existence, uniqueness, regularity, scattering as well as continuous dependence on the initial data. The first result of this type was first proved in [3] in high dimension $n \geq 4$ using the Coulomb gauge and suitable dispersive type estimates for the linear Schrödinger equation. The more difficult lower dimensional case $n = 2$ was proved in [4]. This requires a Schrödinger type counterpart[3] of the null frame spaces, as well as the caloric gauge. The corresponding result for the $\mathbb{H}^2$ target, though not explicitly spelled out in [4], follows by an almost identical argument.

The Schrödinger map counterpart of the large data problem result for wave maps in Theorem 3.7 is still open. However, we have the following partial result:

Theorem 3.9 (Bejenaru-Ionescu-Kenig-Tataru [1],[2])**.** *The following hold for the Schrödinger map equation in dimension $n = 2$ in the* 1-equivariant *class:*

a) For the $\mathbb{H}^2$ target we have global well-posedness and scattering for all large data in the energy space $\dot H^1$.

b) For the $\mathbb{S}^2$ target we have global well-posedness and scattering for all large data in the energy space $\dot H^1$ below the ground state energy.

4. Wave maps

4.1. **Small data heuristics.** Here we outline the main difficulties encountered in the study of the small data problem, and describe the ideas needed to overcome these difficulties. For simplicity we confine ourselves to the most interesting case of dimension two. Some simplifications arise in higher dimension, but the principles remain the same.

[3]Considerably simpler than for wave maps, though.

4.2. **A perturbative set-up.** In a first approximation, suppose that we are trying to view the wave map equation in the extrinsic formulation, namely

wm-ext
$$\Box \phi^i = -\mathcal{S}^i_{jk}(\phi)\partial^\alpha \phi^j \partial_\alpha \phi^k, \qquad \phi(0) = \phi_0, \quad \partial_t \phi(0) = \phi_1 \tag{4.1}$$

as a small perturbation of the constant coefficient wave equation. This will not actually work, but it provides very useful insights. For this we would need two function spaces; one, call it S, for solutions, and a second, call it N, for the nonlinearity. For these spaces we would like to have two estimates:

a) a linear bound,

$$\|\phi\|_S \lesssim \|\phi[0]\|_{\dot H^1 \times L^2} + \|\Box \phi\|_N \tag{4.2}$$

b) an estimate for the nonlinearity,

$$\|N(\phi)\|_N \lesssim \|\phi\|_S, \qquad N(\phi) = \mathcal{S}(\phi)\partial^\alpha \phi\, \partial_\alpha \phi \tag{4.3}$$

Further digesting the estimate for the nonlinearity, it would seem natural to break this into three parts:

b1) The algebra property for S.

b2) The null form bilinear estimate

nfe
$$\|\partial^\alpha \phi\, \partial_\alpha \phi\|_N \lesssim \|\phi\|_S^2. \tag{4.4}$$

b3) The product bound $S \cdot N \to N$.

4.2.1. *The Strichartz norms.* A key ingredient in the study of semilinear wave equations is the Strichartz estimates. Here we can easily incorporate the estimates in the structure of our function spaces by setting, in dimension $n = 2$,

SN-Strich
$$S \subset |D|^{-1} L^\infty L^2 \cap |D|^{-\frac14} L^4 L^\infty, \qquad N \supset L^1 L^2 + |D|^{\frac34} L^{\frac43} L^1 \tag{4.5}$$

However, one sees that the Strichartz estimates cannot suffice to estimate the bilinear expression in (4.4). There are two reasons for that:

(i) The balance of the exponents. This is worst in two dimensions and improves as the dimension increases, up to the point where, in $5+1$ dimensions, it becomes favorable.

(ii) The balance of the derivatives. Because of the form of (4.4), one actually cannot use the full range of Strichartz exponents for each factor. This limitation is independent of the dimension.

Thus, by themselves, Strichartz estimates will not solve the problem. To remedy that, one needs to take advantage of the structure of the nonlinearity.

heuristics

4.2.2. *The null structure.* We denote by τ the time Fourier variable and by ξ the space Fourier variable. We will refer to ξ as the frequency. An important role is played by the null cone $\tau^2 = \xi^2$, which is the characteristic set of $\Box$. The distance to the null cone, which has size $||\tau| - |\xi||$, will be referred to as modulation.

The symbol of the bilinear form $\partial^\alpha \phi\, \partial_\alpha \phi$ is $\tau s - \xi\eta$. As it is easy to see, this symbol vanishes if (τ, ξ) and (s, η) are parallel and located on the null cone. This is what we call the **null condition**. The geometric interpretation of this is that the nonlinear interaction of waves traveling in the same direction is killed in the nonlinearity, leaving the bulk of the nonlinear interaction to come from transversal waves. Heuristically that should be better behaved, because transversal waves have a short interaction time.

As the null condition depends on location of waves in the Fourier space, it cannot be handled via Strichartz estimates, which are invariant with respect to Fourier translations. Instead, one needs to take advantage of the $X^{s,b}$ type structure. The homogeneous $X^{s,b}$ spaces associated to the homogeneous wave equation are defined using the size of the Fourier transform,

$$\|u\|_{X^{s,b}} = \|\hat{u}(\tau,\xi)|\xi|^s||\tau| - |\xi||^b\|_{L^2}$$

Scaling considerations would dictate that we choose

$$S = X^{1,\frac{1}{2}}, \qquad N = X^{0,-\frac{1}{2}}$$

Unfortunately this is just outside the range of indices for which these spaces are well defined.

To avoid the above difficulty one may use the $U^2_\square$ and $V^2_\square$ type spaces associated to the wave equation. These were first introduced in unpublished work of the author in connection to wave maps, and are described in detail elsewhere in these notes. They can be associated separately to each half wave and then combined using suitable multiplier. They are close to the above $X^{s,b}$ spaces, in the sense that

XUV
$$X^{1,\frac{1}{2},\infty} \subset V^2_\square \dot{H}^1 \subset U^2_\square \dot{H}^1 \subset X^{1,\frac{1}{2},1} \tag{4.6}$$

where the third index in the $X^{s,b}$ notation is a Besov index with respect to modulation.

For the moment we neglect what happens far away from the null cone, which will turn out to be easier to deal with anyway. Then one would roughly have to choose

SN-U2
$$S \subset U^2_\square \dot{H}^1, \qquad N \supset DU^2_\square L^2 \tag{4.7}$$

In view of Strichartz type embeddings associated to the U^2 and V^2 spaces, this is stronger than (4.5). With this choice we would have to prove a bound of the type

must-have0
$$\|\partial^\alpha \phi^1 \partial_\alpha \phi^2\|_{DU^2_\square L^2} \lesssim \|\phi^1\|_{U^2_\square \dot{H}^1} \|\phi^2\|_{U^2_\square \dot{H}^1} \tag{4.8}$$

By duality $(DU^2_\square L^2)^* = V^2_\square L^2$ this becomes

-must-have
$$\left| \int \partial^\alpha \phi^1 \partial_\alpha \phi^2 \phi^3 \; dxdt \right| \lesssim \|\phi^1\|_{U^2_\square \dot{H}^1} \|\phi^2\|_{U^2_\square \dot{H}^1} \|\phi^3\|_{V^2_\square L^2} \tag{4.9}$$

To test this theory, we consider the usual Littlewood-Paley trichotomy. In order to be able to work with U^2 atoms, we also neglect for now the difference between V^2L^2 and U^2L^2. Then we can prove the following sharp dyadic estimate:

free-waves
Lemma 4.1. *Assume that $j \leq k$. Then the following dyadic estimates hold:*

$$\left| \int \partial^\alpha \phi^1_k \partial_\alpha \phi^2_j \phi^3_k \; dxdt \right| \lesssim 2^{j+k} \|\phi^1_k\|_{U^2_\square L^2} \|\phi^2_j\|_{U^2_\square L^2} \|\phi^3_k\|_{U^2_\square L^2} \tag{4.10}$$

respectively

$$\left| \int \partial^\alpha \phi^1_k \partial_\alpha \phi^2_k \phi^3_j dxdt \right| \lesssim 2^{\frac{k+3j}{2}} \|\phi^1_k\|_{U^2_\square L^2} \|\phi^2_k\|_{U^2_\square L^2} \|\phi^3_j\|_{U^2_\square L^2} \tag{4.11}$$

Proof. The proof of the lemma is fairly simple. First of all, it suffices to prove the result for U^2 atoms. Secondly, by considering the nesting of the steps in each atom, one sees that it suffices to assume that two of the three atoms are free waves. Remembering the relation between U^2 and $X^{s,b}$ spaces, we are left with having to prove bilinear L^2 estimates for free waves. We need to consider two cases depending on the frequency balance of the two free waves:

a) *high* $\times$ *low* free wave interactions. Denoting by 2^k, respectively 2^j the size of two frequencies, we will prove the estimate

l2w-hl
$$\|\partial^\alpha \phi_j \partial_\alpha \phi_k\|_{X^{0,-\frac{3}{4}}} \lesssim 2^{k+\frac{3}{4}j} \|\phi_k(0)\|_{L^2} \|\phi_j(0)\|_{L^2} \tag{4.12}$$

where the output modulation is at most 2^j. Let ξ, respectively η be the frequencies for the two inputs. The output frequency $\xi+\eta$ will have size 2^k, but we also need to compute its distance d from the null cone. This distance turns out to be related to the angle θ between ξ and η. Precisely, we have

$$2^k d \approx |(\tau+s, \xi+\eta)|_m^2 = 2\langle(\tau,\xi),(s,\eta)\rangle_m \approx \pm 2^{k+j}\theta^2$$

where the sign depends on the relative orientation of the two input cones. Fixing the angle θ between the two waves we can reduce the problem to the following L^2 estimate for two free waves at angle $\theta \in [0,1]$:

l2conv-hl
$$\|\phi_k \phi_j\|_{L^2} \lesssim \theta^{-\frac{1}{2}} 2^{\frac{j}{2}} \|\phi_k(0)\|_{L^2} \|\phi_j(0)\|_{L^2} \tag{4.13}$$

This estimate no longer has anything to do with the curvature of the cone, instead it is based on the *transversality* of the two sectors of the cone. Thus it follows by general principles (see the exposition in [46], though such estimates had been known before, e.g. [20], [8]) since the angle of the two cone sections is θ and the size of the intersection of two translates of them is 2^j.

From here one arrives to (4.12) by adding the size of the symbol of the null form $\tau s - \xi\eta \approx \pm 2^{k+j}\theta^2$. There is an additional orthogonality argument which is needed in order to gain the square summability with respect to θ, but we skip it since it plays no role in the sequel.

a) *high* $\times$ *high* free wave interactions. Denoting by 2^k the size of two input frequencies , and by 2^j the size of the output frequency, we will prove the estimate

l2w-hh
$$\|P_j(\partial^\alpha \phi_k^1 \partial_\alpha \phi_k^2)\|_{X^{0,-\frac{3}{4}}} \lesssim 2^{\frac{1}{2}k+\frac{5}{4}j} \|\phi_k^1(0)\|_{L^2} \|\phi_k^2(0)\|_{L^2} \tag{4.14}$$

where the output modulation is at most 2^j. As before let ξ, respectively η be the Fourier variables for the two inputs. The output frequency $\xi+\eta$ is restricted to a 2^j cube, so by orthogonality we can also restrict ξ and η to 2^j cubes.

This time the distance of $\xi+\eta$ from the null cone is related to the angle θ between ξ and η by the relation

$$2^j d \approx \pm 2^{2k}\theta^2$$

where the sign depends on the relative orientation of the two input cones. Fixing the angle θ between the two waves we can reduce the problem to the following L^2 estimate for two free waves localized in 2^j cubes at frequency 2^k and at angle θ:

l2conv-hh
$$\|\phi_k^1 \phi_k^2\|_{L^2} \lesssim \theta^{-\frac{1}{2}} 2^{\frac{j}{2}} \|\phi_k^1(0)\|_{L^2} \|\phi_k^2(0)\|_{L^2} \tag{4.15}$$

This is again a transversality estimate which follows by general principles. From here (4.14) is obtained by adding the size of the symbol of the null form $2^{2k}\theta^2$. $\square$

Compare the needed bound (4.8) with what is actually proved in Lemma 4.1. On the positive side, we have

- extra gains in the *high* $\times$ *high* $\to$ *low* interactions.
- extra gains at small interaction angles

On the negative side, we have

- possible losses in the transition from U^2 to V^2 in (4.9)
- lack of dyadic summation with respect to low frequencies in *low* $\times$ *high* $\to$ *high* interactions.

Both of these difficulties are nontrivial, and will be successively discussed in what follows.

4.2.3. *The null frame spaces.* As mentioned above, one of the difficulties in the direct approach above is the need to transition from V^2 to U^2 spaces in bilinear estimates. This venue was initially pursued by the author, and, on the positive side, it led to the introduction of the U^p and V^p type spaces to the field of dispersive equations. Unfortunately this attempt was not entirely successful, and a more radical reworking of the function spaces S and N was eventually introduced in [47]. We remark that at this point we do have a well established mechanism for transitioning from V^2 to U^2 spaces in estimates, see [18]. However this transition entails logarithmic frequency losses of one type or another, which seem to be too much for this particular problem.

Backtracking to the proof of the estimates (4.13) and (4.15), the key idea is that one would like to have a version of that which also applies to inhomogeneous waves. We focus on the first bound, and revisit its proof. Rather than thinking of it as a convolution of two surface carried distributions in the Fourier space, of the form, say,

$$\|f_j(\xi)\delta_{\tau=\pm|\xi|} * f_k(\xi)\delta_{\tau=\pm|\xi|}\|_{L^2} \lesssim \theta^{-\frac{1}{2}} 2^{\frac{j}{2}} \|f_j\|_{L^2}\|f_k\|_{L^2} \tag{4.16}$$

where $\hat{\phi}_j = f_j(\xi)\delta_{\tau=\pm|\xi|}$ and $\hat{\phi}_k = f_k(\xi)\delta_{\tau=\pm|\xi|}$, we instead take advantage of the extra dimension that we have available to foliate the frequency μ waves with respect to null rays in frequency,

$$f_j(\xi)\delta_{\tau=\pm|\xi|} = \int_\omega f_j^\omega \, d\omega, \qquad f_j^\omega = f_j(r\omega)\delta_{\tau=\pm|\xi|}\delta_{\xi=r\omega}$$

For each f_j^ω we have the bilinear estimate

$$\|f_j^\omega * f_k(\xi)\delta_{\tau=\pm|\xi|}\|_{L^2} \lesssim \theta^{-1}\|f_\mu(\omega r)\|_{L^2_r}\|f_\lambda\|_{L^2} \tag{4.17}$$

simply due to the fact that the incidence angle is θ^2 (compare this with the angle θ of the two surfaces !). Then (4.16) follows easily from (4.17) by Cauchy-Schwartz with respect to ω after also accounting for the change in the surface measure.

So far all we have is an alternate proof of (4.13). The key observation now is that we can rework the proof of (4.17) in terms of mixed L^p norms as follows. If $\phi_j^\omega = \widehat{f_j^\omega}$, then by Plancherel we have the estimate

$$\|\phi_j^\omega\|_{L^2_\gamma L^\infty_{\gamma^\perp}} = \|f_j(\omega r)\|_{L^2_r}, \qquad \gamma = (\omega, \pm|\omega|)$$

On the other hand, using the fact that ω is at angle θ from the support of f_k, we also have the characteristic energy estimate

$$\|\phi_\lambda\|_{L^\infty_\gamma L^2_{\gamma^\perp}} \approx \theta^{-1}\|f_\lambda\|_{L^2}$$

Then (4.17) follows from the last two relations. This suggests that the space S should include, beside the standard Strichartz norm and the U^2 structure, the following two components associated to null frames:

- characteristic energy norms $\cap_\omega L^\infty_\gamma L^2_{\gamma^\perp}$
- foliated norms $\sum_\omega L^2_\gamma L^\infty_{\gamma^\perp}$

By duality considerations, the space N also needs to include

- dual characteristic energy norms $\sum_\omega L^1_\gamma L^2_{\gamma^\perp}$

Fortunately the second set of dual spaces $\cap_\omega L^2_\gamma L^1_{\gamma^\perp}$ turns out not to be needed.

Both of these have to be introduced carefully, with suitable frequency, modulation and angular localizations. An additional difficulty occurs when applying this idea to *high* $\times$ *high* interactions, where one needs to either allow for radial frequency localizations below the frequency scale, or to admit some losses in the interaction angle or in the high-low frequency balance in the estimates. Fortunately this is not a crucial issue since there is sufficient room there to allow for some flexibility.

heuristics

4.2.4. *The paradifferential equation and renormalization.* Suppose now that we have good function spaces S and N for which the dyadic version of the null form estimates holds,

-must-S-hl

$$\|\partial^\alpha \phi_j \partial_\alpha \phi_k\|_N \lesssim \|\phi_j\|_S \|\phi_k\|_S, \qquad j < k \tag{4.18}$$

ust-have0a

$$\|P_j(\partial^\alpha \phi_k \partial_\alpha \phi_k)\|_N \lesssim 2^{-\delta|j-k|} \|\phi_k\|_{S^1} \|\phi_k\|_{S^1}, \qquad j \lesssim k \tag{4.19}$$

While the second has some extra room, the first one is tight, and does not allow for a favorable j summation since the norms of ϕ_j are only square summable. This suggests that the nonlinearity in the wave map equation is actually nonperturbative. If that is the case, then the next best thing to do is to understand exactly what is the nonperturbative part. That immediately leads to the paradifferential formulation of the problem, namely

$$\Box \phi^i_k = -2\mathcal{S}^i_{jl}(\phi)_{<k} \partial^\alpha \phi^j_{<k} \partial_\alpha \phi^l_k + perturbative(N)$$

One advantage in doing this is that now we only need to study a linear equation, where the coefficients have lower frequency. The above equation is closely linked to the linearized wave map equation; indeed, it largely represents a high frequency linearized wave evolving on a low frequency background.

A generic equation of the above form does not seem to have enough structure to allow for good linear estimates. However, so far we have not used at all the geometry of the problem. To take advantage of that we begin with the orthogonality relation

$$\mathcal{S}^l_{ji}(\phi) \partial_\alpha \phi^l = 0$$

Transitioning to the paradifferential form of this and combining it with the previous paradifferential equation we arrive at a more favorable equation,

para-anti

$$\Box \phi_k = -2A_i(\phi)_{<k} \partial^\alpha \phi^i_{<k} \partial_\alpha \phi_k + perturbative(N) \tag{4.20}$$

where the matrices $(A_i)^j_l = \mathcal{S}^j_{il} - \mathcal{S}^l_{ij}$ are antisymmetric. This antisymetry adds some conservation structure to the paradifferential equation; this is closely linked to the question of getting good energy estimates for solutions to (4.20).

Tao [45]'s approach to the above equation in the $\mathbb{S}^2$ case was to develop a renormalization procedure which transforms the nonlinearity into a a perturbative nonlinearity in the context of the null frame spaces. This is reminiscent of Helélein's work on harmonic maps, and is achieved in a multiplicative way in the paradifferential setting. Precisely, one seeks a linear transformation

$$w_k = \mathcal{O}_{<k} \psi_k$$

which transforms the previous equation into the flat wave equation

$$\Box w_k = perturbative(N) \tag{4.21}$$

In the context of the frame method introduced earlier, this corresponds to studying high frequency solutions to the linearized wave map equation, represented in a favorable frame in the tangent space TM.

Substituting into the equation and neglecting some lower order terms, one sees that this works provided that the (orthogonal or almost orthogonal) matrix valued function $\mathcal{O}_{<k}$ is a reasonably good approximate solution for the system of equations

$$\partial_\alpha \mathcal{O}_{<k} = (A_i)_{<k} \partial_\alpha \phi^i_{<k}$$

The construction of such a renormalization matrix $\mathcal{O}$ is a key idea of Tao [45]. This construction was further refined and simplified in Tataru [48] and later in Sterbenz-Tataru [37]. One choice that needs to be made here is between the frequency localization and the orthogonality of $O_{<k}$; both are desirable but seem mutually exclusive. Frequency localization is easier to work with and was the preferred choice in the small data problem in [45], [48]. However, for large data the orthogonality losses become unmanageable, and instead one must sacrifice frequency localization, see [37].

An alternate approach, based on the frame method with the Coulomb gauge, was developed by Krieger [22] for the case of an $\mathbb{H}^2$ target.

4.3. **Function spaces.** Here we define the function spaces S and N, following Sterbenz-Tataru [37]. The space N is essentially as originally introduced in Tataru [47]; there the space $\Box^{-1}N$ was used in place of S, along with the key embedding $\Box^{-1}N \subset S$. Tao [45] observed that using S instead of $\Box^{-1}N$ as the main function space helps with the algebra type properties. Tao's version of S was then strengthened to some extent in Sterbenz-Tataru [37]. A related but somewhat different modification of S was proposed by Krieger [22].

We recall that P_k denote Littlewood-Paley localization with respect to the spatial frequency. For modulation localizations we use the space-time multipliers Q_j with symbol:

$$q_j(\tau,\xi) \ = \ \varphi\big(2^{-j}\big||\tau| - |\xi|\big|\big) \ ,$$

where φ truncates smoothly on a unit annulus. We denote by $Q_j^{\pm}$ the restriction of this multiplier to the upper or lower time frequency space.

Beside the frequency and modulation decompositions, we also need to deal with the angular decompositions which are needed for the proof of the bilinear estimates. We denote by $\kappa \in K_l$ a collection of caps of diameter $\sim 2^{-l}$ providing a finitely overlapping cover of the unit sphere. According to this decomposition, we cut up the spatial frequency domain according to:

$$P_k \ = \ \sum_{\kappa \in K_l} P_{k,\kappa} \ .$$

These decompositions usually occur in conjunction with modulation cutoffs up to 2^j where $j = k - 2l$. This is related to the discussion in Section 4.2.2; another interpretation of this scale choice is that it corresponds to the thinnest angular slabs of angle 2^{-l} on the null cone which are well approximated by a parallelipiped, i.e. have no curvature.

For each integer k we define the following frequency localized norm:

$$\|\phi\|_{S_k} \ := \ \|\nabla_{t,x}\phi_k\|_{L^\infty_t(L^2_x)} + \|\nabla_{t,x}\phi_k\|_{X^{0,\frac12}_\infty} + \|\phi_k\|_{\underline{S}} + \sup_{j<k-20} \|\phi\|_{S[k;j]} \ . \tag{4.22}$$

with components as follows:

- The fixed frequency space $X_p^{s,b}$ is defined as:

$$\| P_k\phi \|^p_{X_p^{s,b}} \ := \ 2^{psk} \sum_j 2^{pbj} \| Q_j P_k \phi \|^p_{L^2_t(L^2_x)} \ ,$$

  with the obvious definition for $X_\infty^{s,b}$.

- The "physical space Strichartz" norms are given by

phys_str
$$\| \phi_k \|_{\underline{S}} \ := \ \sup_{(q,r):\ \frac{2}{q}+\frac{1}{r}\leqslant\frac{1}{2}} 2^{(\frac{1}{q}+\frac{2}{r}-1)k} \| \nabla_{t,x}\phi_k \|_{L^q_t(L^r_x)} \ , \tag{4.23}$$

- The "modulational Strichartz" norms are

_Str_space
$$\| \phi \|_{S[k;j]} \ := \ \sup_\pm \Big(\sum_{\kappa\in K_l} \| Q^\pm_{<k-2l} P_{k,\pm\kappa}\phi \|^2_{S[k,\kappa]} \Big)^{\frac{1}{2}} \ , \qquad l = \frac{k-j}{2} > 10 \ , \tag{4.24}$$

- The "angular Strichartz" space is defined in terms of the three components:

str_norm
$$\begin{aligned} \| \phi \|_{S[k,\kappa]} \ := \ & 2^k \sup_{\omega\notin 2\kappa} \mathrm{dist}(\omega,\kappa) \| \phi \|_{L^\infty_{t_\omega}(L^2_{x_\omega})} + 2^k \| \phi \|_{L^\infty_t(L^2_x)} \\ & + 2^{\frac{1}{2}k} |\kappa|^{-\frac{1}{2}} \inf_{\sum_\omega \phi^\omega = \phi} \sum_\omega \| \phi^\omega \|_{L^2_{t_\omega}(L^\infty_{x_\omega})} \ . \end{aligned} \tag{4.25}$$

The first component on the RHS above will often be referred to as $N\!\!\!/FA^*$.

We define S as the space of functions ϕ in $\mathbb{R}^{2+1}$ with $\nabla_{x,t}\phi \in C(\mathbb{R}; L^2_x)$ and finite norm:

$$\| \phi \|^2_S \ = \ \| \phi \|^2_{L^\infty_t(L^\infty_x)} + \sum_k \| \phi \|^2_{S_k} \ ,$$

Two other norms related to S play an auxiliary role in the study of the large data problem, namely

- The null frame energy:

uE_norm
$$\| \phi \|_{\underline{E}} \ := \ \| \nabla_{t,x}\phi \|_{L^\infty_t(L^2_x)} + \sup_\omega \| \not\nabla^\omega_{t,x}\phi \|_{L^\infty_{t_\omega}(L^2_{x_\omega})} \ , \tag{4.26}$$

- The high modulation L^2 norm:

uX_norm
$$\| \phi \|_{\underline{X}_k} \ := \ 2^{-\frac{1}{2}k} \| \Box P_k \phi \|_{L^2_t(L^2_x)} \ . \tag{4.27}$$

We also define $\underline{X}$ as the square sum of $\underline{X}_k$. Notice that there are no square sums or frequency localizations in the norm $\underline{E}$. This makes proving $\underline{E}$ bounds amenable to energy estimates techniques, bypassing the more difficult bilinear and multilinear estimates. The $\underline{X}$ bounds are also easier to obtain and provide stronger high modulation bounds than what is included in the S norm.

In the same manner as in the case of the S space, for each integer k we define the dyadic versions of the N norm by

N_norm
$$\begin{aligned} \| F \|_{N_k} \ := \ & \inf_{F_A+F_B+\sum_{l,\kappa} F_C^{l,\kappa} = F} \Big(\| P_k F_A \|_{L^1_t(L^2_x)} + \| P_k F_B \|_{X_1^{0,-\frac{1}{2}}} \\ & + \sum_\pm \sum_{l>10} \Big(\sum_\kappa \inf_{\omega\notin 2\kappa} \mathrm{dist}(\omega,\kappa)^{-2} \| Q^\pm_{<k-2l} P_{k,\pm\kappa} F_C^{l,\kappa} \|^2_{L^1_{t_\omega}(L^2_{x_\omega})} \Big)^{\frac{1}{2}} \Big) \ . \end{aligned} \tag{4.28}$$

We will often refer to the last component on the RHS above as $N\!F\!A$, and the norm applied to a fixed $Q^{\pm}F_C^{l,\kappa}$ as $N\!F\!A[\pm\kappa]$.

The full N norm is

$$\|F\|_N^2 = \sum_k \|P_k F\|_{N_k}^2$$

All of these spaces have versions which are restricted to time intervals I, denoted e.g. by $S[I]$, respectively $N[I]$. Since the interval truncation does not commute with the time Fourier transform, some minor technical issues arise in the process. These are skipped here.

4.3.1. *Frequency envelopes.* In many places of the subsequent analysis involving the S spaces it pays to keep a more careful track of how much of the S norm of wave maps is concentrated at various frequencies. This is conveniently expressed in the language of frequency envelopes.

A sequence c_k is called a frequency envelope for ϕ in S if the following three requirements are satisfied:

- Norm control:
$$\|\phi_k\|_S \leq c_k$$
- Norm equivalence:
$$\sum c_k^2 \approx \sum \|\phi_k\|_S^2$$
- Slowly varying:
$$|c_j/c_k| \lesssim 2^{\delta|j-k|}$$
for a fixed small universal constant δ.

A similar terminology is used with respect to all of the other norms in the paper, e.g. the initial data space $\dot H^1 \times L^2$, the space N, etc.

4.3.2. *Linear analysis in the S and N spaces.* The linear component of our estimates has the form

Proposition 4.2. *The following estimate holds for functions which are localized at frequency 2^k:*

nk-to-sk

$$\|\phi_k\|_{S_k} \lesssim \|\phi_k[0]\|_{\dot H^1\times L^2} + \|\Box\phi_k\|_{N_k} \tag{4.29}$$

Outline of the proof. The proof is relatively straightforward when interpreted in terms of the U^2 norms. Set $F = \Box\phi_k$. With notations as in the above definition of the N_k norm, consider first the case when $F = F_A + F_B$. By Strichartz type embeddings and the dual to (4.6) it is fairly easy to see that $F \in DU^2_\Box L^2$, therefore the corresponding solution ϕ_k belongs to $U^2_\Box \dot H^1$, so it remains to show that $U^2_\Box \dot H^1 \subset S$. The first and third components of the S norm are easy to estimate via bounds for free waves and then for atoms. The third component of the S norm is bounded by (4.6). It remains to consider the $S[k,j]$ norms. The U^2 space is well behaved with respect to frequency and modulation localizations,

$$\sum_{\kappa\in K_l} \|\, Q^{\pm}_{<k-2l} P_{k,\pm\kappa}\phi_k \,\|^2_{U^2_\Box \dot H^1} \lesssim \|\phi_k\|^2_{U^2_\Box \dot H^1}$$

so it remains to estimate the $S[k,\kappa]$ norm for each localized piece. But this is easily done again by starting with the known bounds free solutions, which are then transferred to $U^2 H^1$ atoms.

Lastly, consider the case when $F = \sum_{l,\kappa} F_C^{l,\kappa}$. On one hand we can place $F_C^{l,\kappa}$ in $DV_\Box^2 L^2$, which follows by duality from the embedding of $U_\Box^2 \dot H^1$ into the NFA^* component of the S_k space. On the other hand we can place $F_C^{l,\kappa}$ into $DU_\omega^2 L^2$, which is the U^2 space corresponding to the wave evolution in the null direction associated to ω.

Thus, denoting by $\phi = \sum \phi^{l,\kappa}$ where $\phi^{l,\kappa}$ is the solution to

$$\Box \phi^{l,\kappa} = F_C^{l,\kappa}, \qquad \phi^{l,\kappa}[0] = 0,$$

we can first bound $\phi^{l,\kappa}$ in $V_\Box^2 \dot H^1$. By frequency orthogonality this leads to a $V_\Box^2 \dot H^1$ for ϕ, and this suffices for the first three components of the S_k norm.

Secondly, we can bound $\phi^{l,\kappa}$ in $U_\omega^2 \dot H^1$ with $\omega = \omega(\kappa)$. Now to estimate the $S[k,j]$ norm of ϕ we consider two cases:

(i) $l > l' = \frac{j-k}{2}$. Then each $\phi^{l,\kappa}$ is measured with respect to a collection of $S[k,\kappa']$ norms with $\kappa' \in K_{l'}$. We can argue separately for each ω that $U_\omega^2 \dot H^1 \subset S[k,j]$ and then use the square summability with respect to κ to sum up the results.

(ii) $l < l' = \frac{j-k}{2}$. Then each corresponding $S[k,\kappa']$ norm applies to a collection of $\phi^{l,\kappa}$. For the first two components of the $S[k,\kappa']$ norm, we estimate them directly for each $\phi^{l,\kappa}$, and then use L^2 orthogonality based on the frequency/modulation localization to add them up. For the last component of the $S[k,\kappa']$ norm, we simply sum up the bounds for each $\phi^{l,\kappa}$; orthogonality does not hold, but it is also not needed. $\Box$

4.3.3. *Multilinear estimates.* For the nonlinear side of our problem we need not only the bilinear null form estimate described earlier, but also additional bounds which account for the role of the $\mathcal{S}(\phi)$ factor. To start with, we have:

bi-tri **Proposition 4.3.** *The following bilinear and trilinear estimates hold for the S and N spaces:*

- *Product estimates:*

S_prod2
$$\| \phi^{(1)}_{<k+O(1)} \cdot \phi^{(2)}_k \|_S \lesssim \| \phi^{(1)}_{<k+O(1)} \|_S \cdot \| \phi^{(2)}_k \|_S , \tag{4.30}$$

S_prod3
$$\| P_k(\phi^{(1)}_{k_1} \cdot \phi^{(2)}_{k_2}) \|_S \lesssim 2^{-(\max\{k_i\}-k)} \| \phi^{(1)}_{k_1} \|_S \cdot \| \phi^{(2)}_{k_2} \|_S , \tag{4.31}$$

_prod_est1
$$\| P_k(\phi_{<k+O(1)} \cdot F_k) \|_N \lesssim \| \phi \|_S \cdot \| F_k \|_N , \tag{4.32}$$

_prod_est2
$$\| P_k(\phi_{k_1} \cdot F_{k_2}) \|_N \lesssim 2^{-\delta(k-k_2)_+} \| \phi_{k_1} \|_S \cdot \| F_{k_2} \|_N , \tag{4.33}$$

- *Bilinear Null Form Estimates:*

_L2_est_bi
$$\|P_k\big(\partial^\alpha \phi^{(1)}_{k_1} \cdot \partial_\alpha \phi^{(2)}_{k_2}\big)\|_{L^2_t(L^2_x)} \lesssim 2^{\frac12 \min\{k_i\}} 2^{-(\frac12+\delta)(\max\{k_i\}-k)} \prod_i \|\phi^{(i)}_{k_i}\|_S , \tag{4.34}$$

ard_est_bi
$$\| P_k(\partial^\alpha \phi^{(1)}_{k_1} \cdot \partial_\alpha \phi^{(2)}_{k_2}) \|_N \lesssim 2^{-\delta(\max\{k_i\}-k)} \prod_i \| \phi^{(i)}_{k_i} \|_S . \tag{4.35}$$

- *Trilinear Null Form Estimate:*

rd_est_tri
$$\| P_k(\phi^{(1)}_{k_1} \cdot \partial^\alpha \phi^{(2)}_{k_2} \cdot \partial_\alpha \phi^{(3)}_{k_3}) \|_N \lesssim 2^{-\delta(\max\{k_i\}-k)} 2^{-\delta(k_1-\min\{k_2,k_3\})_+} \prod_i \| \phi^{(i)}_{k_i} \|_S. \tag{4.36}$$

The bilinear estimates are essentially the dyadic counterparts of the bounds discussed in Section null_heuristics 4.2.2. The last trilinear estimate provides a key improvement over the composition of bilinear bounds, which plays a major role in the renormalization procedure in Section renorm_heuristics 4.2.4. The proofs follow largely from the principles discussed in Section null_frames 4.2.3, and are omitted;

instead we refer the reader to [47], [45] and [37]. As a consequence of the above results we have

Proposition 4.4. *a) The space S is an algebra, and the following Moser type estimates hold for any bounded function G with uniformly bounded derivatives:*

$$\| G(\phi) \|_S \lesssim \| \phi \|_S (1 + \| \phi \|_S^3) , \tag{4.37}$$

In addition, if c_k is a frequency envelope for ϕ, then

$$\| G(\phi)_k \|_S \lesssim (1 + \| \phi \|_S^3) c_k$$

b) The product estimate $S \times N \to N$ holds.

Outline of proof. The nontrivial part of the proposition is the Moser estimate. For that, following [48], we use multilinear paradifferential decompositions. For $h \in \mathbb{R}$ we can write

$$\frac{d}{dh} F(\phi_{<h}) = \phi_h F'(\phi_{<h})$$

or in integral form

$$F(\phi) = F(\phi_{<l}) + \int_l^\infty \phi_h F'(\phi_{<h}) dh \tag{4.38}$$

This suffices for energy estimates, but not for estimates in the S type spaces. Hence we iterate this computation to obtain

$$\begin{aligned} F(\phi) &= F(\phi_{<l}) + F'(\phi_{<l}) \int_l^\infty \chi(h) \phi_h dh + F''(\phi_{<l}) \int_{[l,\infty)^2} \chi(h) \phi_{h_0} \phi_{h_1} dh \\ &+ \int_{[l,\infty)^3} \chi(h) \phi_{h_0} \phi_{h_1} \phi_{h_2} F'''(\phi_{<h_2}) dh \end{aligned} \tag{4.39}$$

where by $\chi(h)$ we denote the ordering function

$$\chi(h) = 1_{h_j \leq h_{j-1} \leq \cdots \leq h_0}.$$

This expansion allows us to successively build estimates for $F(\phi_{<l})$ as follows:

(i) First, by direct differentiation, we have

$$\|\nabla F(\phi_{<k})\|_{L^\infty} \lesssim 2^k, \qquad \|\nabla F(\phi_{<k})\|_{L^\infty L^2} \lesssim 1$$

(ii) Next, repeated differentiation followed by Littlewood-Paley projections yields the high frequency decay

$$\|P_j F(\phi_{<k})\|_{L^\infty} \lesssim 2^{N(k-j)}, \qquad \|P_j F(\phi_{<k})\|_{L^\infty L^2} \lesssim 2^{-k+N(k-j)}, \qquad j > k$$

(iii) Applying (4.39) to $\phi_{<k}$ and letting $l \to -\infty$ the first term drops, and using the Strichartz estimates[4] for ϕ and the bounds in the previous steps we obtain the better high frequency decay

$$\|P_j F(\phi_{<k})\|_{L^2} \lesssim 2^{-\frac{3k}{2} + N(k-j)}, \qquad j > k + 10$$

For all practical purposes this allows us to assume that $F(\phi_{<k})$ is localized at frequency $\lesssim 2^k$; the contributions of higher frequencies are easier to estimate.

(iv) To estimate a component of the S_k norm of $F(\phi)$ which involves a modulation truncation at modulation $2^j < 2^k$, we apply (4.39) to ϕ with $l = j - 10$. The factors $F'(\phi_{<l})$ and

[4] It takes exactly three Strichartz estimates to place a product in L^2.

$F''(\phi_{<l})$ are bounded so they preserve all mixed L^p norm, without affecting the frequency localization (except for better behaved tails). In the last term in (4.39) we have the higher frequency factor $F'''(\phi_{<h_2})$. However, this is combined with a trilinear expression $\phi_{h_0}\phi_{h_1}\phi_{h_2}$ which by Strichartz and multilinear S estimates has an L^2 structure on the 2^{h_2} frequency scale; hence, it again suffices to use the L^∞ bound for $F'''(\phi_{<h_2})$.

□

4.4. **Renormalization.** The idea behind the renormalization is to consider a linear paradifferential equation of the type

$$(\Box + 2A_i(\phi)_{<k-m}\partial^\alpha \phi^i_{<k-m}\partial_\alpha)\psi_k = F_k$$

with antisymmetric A_i's, and to obtain estimates of the type

$$\|\psi_k\|_S \lesssim \|\psi_k[0]\|_{\dot H^1\times L^2} + \|F_k\|_N. \tag{4.40}$$

Here m is a large parameter which depends on the S size of ϕ in the coefficients; it is essential in the large data problem, but it plays no role for small data.

The strategy is to use a renormalization matrix $O_{<k-m}$ to perform a change of variable $w_k = O_{<k-m}\psi_k$ so that the equation for w_k is

$$\Box w_k = O_{<k-m}F_k + error(N)$$

To motivate the choice of O we compute the above error,

$$\begin{aligned} error = &\ \left(\Box O_{<k-m} - O_{<k-m}(\Box + 2A_i(\phi)_{<k-m}\partial^\alpha\phi^i_{<k-m}\partial_\alpha)\right)\phi_k \\ = &\ \Box O_{<k-m}\phi_k + 2(\partial^\alpha O_{<k-m} - O_{<k-m}A_i(\phi)_{<k-m}\partial^\alpha\phi^i_{<k-m})\partial_\alpha\phi_k \end{aligned}$$

The first term in the error is in some[5] sense better behaved because both derivatives apply to the lower frequency factor. In the second term, in view of the trilinear estimate (4.36), we can neglect the terms where the frequency of A_i is comparable or larger than the frequency of ϕ^i. Hence, defining

$$B_k = A_i(\phi)_{<k-C}\phi^i_k$$

a reasonable choice would be to select O_k so that

$$O_k = O_{<k}B_k \tag{4.41}$$

Since O_k is continuously interpreted as $\partial_k O_{<k}$, it follows that $O_{<k}$ is defined as the solution to the ode

$$\frac{d}{dk}O_{<k} = O_{<k}B_k, \qquad \lim_{k\to-\infty} O_{<k} = I_m \tag{4.42}$$

Defining O_k as such has one key advantage, namely that the antisymmetry of B_k insures that $O_{<k}$ remains an orthogonal matrix, and provides good L^p type bounds for its derivatives. There is also a significant disadvantage, namely that the frequency localization is lost; fortunately the frequency tails turn out to decrease rapidly. The bounds for $O_{<k}$ are summarized as follows:

propphiu **Proposition 4.5.** *Let ϕ be a wave-map with energy E and S norm F and S frequency envelope $\{c_k\}$. Then the orthogonal matrix $O_{<k}$ defined above and its k derivative O_k have the following properties:*

[5]This still has to be proved once $O_{<k}$ is constructed, and it it not entirely straightforward.

- *(S_k bounds for O_k) Each O_k obeys the bounds:*

env_est1
$$\| P_{k'} O_k \|_S \lesssim_F 2^{-\delta|k-k'|} 2^{-C(k'-k)_+} c_k \ , \tag{4.43}$$

env_est1h
$$\| P_{k'} \nabla^J_{t,x} O_k \|_{L^1_t(L^1_x)} \lesssim_F 2^{(|J|-3)k} 2^{-C(k'-k)} c_k \ , \quad k' > k+10, \quad |J| \leqslant 2 \ , \tag{4.44}$$

env_est3
$$\| P_{k'} \big(O_{<k-20} \cdot G_k \big) \|_N \lesssim_F 2^{-|k'-k|} \| G_k \|_N \ , \tag{4.45}$$

env_est2
$$\| P_k \big(\Box O_{k_1} \cdot \psi_{k_2} \big) \|_N \lesssim_F 2^{-|k-k_2|} 2^{-\delta(k_2-k_1)} c_{k_1} \| \psi_{k_2} \|_S \ , \quad k_1 < k_2 - 10 \ . \tag{4.46}$$

- *(The Matrix O Approximately Renormalizes $A_\alpha = \nabla_\alpha B$) We have the formula:*

enorm_form
$$O^\dagger_{<k} \nabla_\alpha O_{<k} \ = \ \nabla_\alpha B_{<k} - \int_{-\infty}^{k} \big[B_{k'}, O^\dagger_{<k'} \nabla_\alpha O_{<k'} \big] \, dk' \ . \tag{4.47}$$

Proof. The main difficulty in the proof is that, since B_k are not small, it is not possible to directly bootstrap the estimates for O_k. Instead the proof is by direct arguments, iterating separately the various components of the S norm, in the following order:

- L^∞ and $L^\infty L^2$ bounds
- Strichartz bounds
- High modulation bounds (i.e. L^2 bounds for $\Box U_k$)
- High frequency bounds (i.e. the estimate (env_est1h 4.44))
- S norm bounds.

Here each step is carried out based on the previous steps, without bootstrapping. The most difficult part, i.e. the S bound, is obtained by using iterated expansions akin to the proof of the Moser estimates. For further details we refer the reader to [MR2657817 37].

□

The main use of the renormalization matrix $O_{<k}$ is in the proof of the $N \to S$ estimates for the paradifferential equation:

p:para
Proposition 4.6 (Gauge Covariant S Estimate)**.** *Let $\psi_k = P_k \psi$ be a solution to the linear problem:*

ced_lin_eq
$$\Box \psi_k \ = \ -2 A^\alpha_{<k-m} \partial_\alpha \psi_k + G_k \ , \tag{4.48}$$

where $\mathcal{A}^\alpha_{<k-m}$ is the $\mathfrak{so}(N)$ matrix:

red_con
$$(\mathcal{A}^\alpha_{<k-m})^a_b \ = \ \big(\mathcal{S}^a_{bc}(\phi) - \mathcal{S}^b_{ac}(\phi) \big)_{<k-m} \partial^\alpha \phi^c_{<k-m} \ . \tag{4.49}$$

Assume that ϕ is a smooth Wave-Map on I with the bounds:

_tdphi_est
$$\| \phi \|_{\underline{E}[I]} + \| \phi \|_{\underline{X}[I]} + \| \phi \|_{S[I]} \ \leqslant \ F \ . \tag{4.50}$$

Furthermore, assume that $m \geqslant m(F) > 20$, for a certain function $m(F) \sim \ln(F)$. Then we have the estimate:

arized_est
$$\| \psi_k \|_S \ \lesssim_F \ \| \psi_k[0] \|_{\dot{H}^1 \times L^2} + \| G_k \|_N \ . \tag{4.51}$$

We remark on the role of the parameter m. If ϕ is small (i.e. F is small in the theorem) then any $m \geq 10$ suffices. However, if ϕ is large, then we need an alternate source for smallness.

Outline of the proof. The proof of this result comes in two flavors:

a) Small ϕ. Set $m = 10$. A direct use of the renormalization matrix $O_{<k-m}$, as shown in the previous section, reduces the problem to an equation for $w_k = O_{<k-m}\psi_k$, namely

$$\Box w_k = O_{<k-m} F_k + Err\ \psi_k$$

where the terms on the right are estimated directly using the bilinear and trilinear estimates in Proposition 4.3,

$$\|O_{<k-m} F_k\|_N \lesssim \|F_k\|_N, \qquad \|Err\ \psi_k\|_N \lesssim F \|\psi_k\|_S$$

The smallness of F yields the smallness of the error term, therefore one can conclude using the $N \to S$ estimate (4.29) for the $\Box$ equation.

b) Large ϕ. In this case the previous argument no longer works because the errors are no longer small. This is where the large parameter m plays a key role, and provides a more subtle form of smallness which replaces the smallness coming from ϕ.

In the first step we consider energy estimates. Precisely, our paradifferential equation is essentially a covariant wave equation, therefore energy estimates can be established directly using integration by parts, with an error which is small for large m. In addition, characteristic energy estimates, i.e. bounds for the $\underline{E}$ norm, are just as easy to obtain. Precisely, we have

$$\|\psi\|_{\underline{E}} \lesssim_F \|\psi[0]\|_{\dot H^1 \times L^2} + 2^{-\delta m} \|\psi_k\|_S$$

In a second step we apply the renormalization procedure; however, instead of directly applying the bilinear and trilinear estimates in Proposition 4.3, we refine them so that the bulk of the error is estimated using the characteristic energy estimates, and only a small part, corresponding to small angle interactions, is done using the full S norm of ψ_k,

$$\|Err\ w_k\|_N \lesssim_F \epsilon^{-N} \|\psi_k\|_{\underline{E}} + \epsilon \|\psi_k\|_S, \qquad \epsilon \ll 1$$

Combining the last two estimates, we obtain (with a new $\delta > 0$)

$$\|Err\ w_k\|_N \lesssim_F \|\psi[0]\|_{\dot H^1 \times L^2} + 2^{-\delta m} \|\psi_k\|_S$$

and now we can use again (4.29) to close the argument provided that m is large enough (depending on F).

We note that all implicit constants are polynomial in F, which leads to a logarithmic dependence of $m(F)$ on F.

$\Box$

4.5. **The small data result.** Here we outline the proof of the small data result in Theorem 3.6. This is achieved in several steps:

4.5.1. *The a-priori estimate.* The aim here is to start with a smooth wave map on a time interval I, which is a-priori assumed to satisfy the bound

$$\|\phi\|_S \le \epsilon \tag{4.52}$$

for some sufficiently small ϵ. Then we establish the following two estimates:

main-wm

$$\|\phi\|_S \lesssim \|\phi[0]\|_{\dot H^1 \times L^2} \tag{4.53}$$

$$\|\phi\|_{S^N} \lesssim \|\phi[0]\|_{\dot H^N \times \dot H^{N-1}} \tag{4.54}$$

where S^N stands for functions with $N-1$ spatial derivatives in S.

In effect it is very convenient to provide a more precise version of this, expressed in the language of frequency envelopes. Precisely, one starts with a frequency envelope c_k for the initial data $\phi[0]$, i.e.

$$\|\phi_k[0]\|_{\dot H^1\times L^2} \leq c_k.$$

Then the estimate to prove is

$$\|\phi_k\|_S \lesssim c_k \tag{4.55}$$

A similar analysis can be carried out at the level of the S^N norms.

To achieve this we begin with the full equation

$$\Box \phi^i = -\mathcal{S}^i_{jl}(\phi)\partial^\alpha \phi^j \partial_\alpha \phi^l$$

apply Littlewood-Paley projections and rewrite it in the paradifferential form

$$(\Box + 2A_j(\phi)_{<k}\partial^\alpha \phi^j_{<k}\partial_\alpha)\phi_k = F_k$$

The functions F_k contain all the interactions not included in the left, an can be estimated directly using the bilinear and trilinear estimates in Proposition 4.3,

$$\|F_k\|_{N_k} \lesssim d_k \|\phi\|_S F(\|\phi\|_S)$$

where d_k is a frequency envelope for ϕ in S, for now unrelated to c_k.

Applying Proposition 4.6 we obtain the bound

$$\|\phi_k\|_S \lesssim \|\phi_k[0]\|_{\dot H^1\times L^2} + \|F_k\|_N$$

which leads to

$$d_k \lesssim c_k + \epsilon d_k$$

Given our assumption on the smallness of ϵ, we obtain $d_k \lesssim c_k$, and the desired conclusion follows.

4.5.2. *Global existence and regularity.* Consider a smooth initial data set $\phi[0]$ with small energy, say $\ll \epsilon$. Then for a short time there is a smooth solution ϕ, which can be easily shown it is small in S. We consider the set of times T for which a smooth solution satisfying $\|\phi\|_{S[-T,T]} \leq \frac12\epsilon$ exists in $[-T,T]$. The family of rescaled functions $\phi(t/T, x/T)$ depends smoothly on T, so it will have an S norm depending continuously on T. By Step 1 it follows that the threshold $\frac12\epsilon$ is never reached. By an open/close argument this shows that the solution exists for all t, and satisfies the bound $\|\phi\|_{S[-T,T]} \leq \frac12\epsilon$.

4.5.3. *Weak Lipschitz dependence on the initial data.* Here we consider the linearized wave map equation, which has the form

lin
$$\Box \psi^l = -(\partial_m \mathcal{S}^l_{ij})(\phi)\psi^m \partial^\alpha \phi^i \partial_\alpha \phi^j - 2\mathcal{S}^l_{ij}(\phi)\partial^\alpha \phi^i \partial_\alpha \psi^j \tag{4.56}$$

The function ψ must satisfy the compatibility condition

compat
$$\psi(t,x) \in T_{\phi(t,x)}M \tag{4.57}$$

Understanding the behavior of these equations is the key to comparing different solutions of the wave maps equation.

The goal here is to show that under the assumption $\|\phi\|_S \leq \epsilon$, we have bound of the form

$$\|\psi\|_{S^{-\delta}} \lesssim \|\psi[0]\|_{\dot H^{1-\delta}\times \dot H^{-\delta}} \tag{4.58}$$

for some small δ.

The proof of this bound is similar to the proof of the main estimate (4.53). We write the equations for ψ_k, which evolve along the same paradifferential flow as the equations for ϕ_k, and then show that the errors are small and use Proposition 4.6.

A consequence of the above bound is an estimate for the difference of solutions,

$$\|\phi_1 - \phi_2\|_{S^{-\delta}} \lesssim \|\phi_1[0] - \phi_2[0]\|_{\dot H^{1-\delta}\times \dot H^{-\delta}} \tag{4.59}$$

4.5.4. *Rough solutions and continuous dependence on the initial data.* Given any small energy data $\phi[0]$, we approximate it with a sequence of regularized data

$$\phi_n[0] \to \phi[0] \qquad in \ \ (\dot H^1 \times L^2) \cap (\dot H^{1-\delta} \times \dot H^{-\delta})$$

It is not difficult to show that $\phi_n[0]$ can be chosen to inherit the frequency envelope from $\phi[0]$. Then we have a corresponding sequence $\phi^{(n)}$ of smooth solutions, which by the previous step is Cauchy in $S^{-\delta}$. It also has a common frequency envelope in S. Together these two facts show that $\phi^{(n)}$ is actually Cauchy in S; thus we obtain a unique limit ϕ which is small in S.

The same argument yields continuous dependence on the initial data in the $(\dot H^1 \times L^2) \cap (\dot H^{1-\delta} \times \dot H^{-\delta})$ topology. Due to the finite speed of propagation, a localized form of this result is also available; it asserts continuous dependence on the initial data in the $H^1_{loc} \times L^2_{loc}$ topology.

4.6. **Energy dispersion.** Here we discuss the first step toward the study of the large data problem. The idea is that there should be some dichotomy between concentration of wave maps and the global existence of large data solutions. In other words, it would be reasonable to expect that if no concentration occurs then solutions persist globally. This was the viewpoint adopted in Sterbenz-Tataru [37].

The interesting question though is what is the meaning of "*concentration*". To address that, in [37] was introduced the notion of energy dispersion. For a time interval I we set

$$\|\phi\|_{ED[I]} = \sup_k \| P_k \phi \|_{L^\infty_{t,x}[I\times\mathbb{R}^2]} \tag{4.60}$$

Then the main result asserts that energy dispersed solutions are good:

Theorem 4.7 (Energy Dispersed Regularity Theorem [37])**.** *There exist two functions*

$$1 \ll F(E), \qquad 0 < \epsilon(E) \ll 1$$

of the energy such that the following statement is true. If ϕ is a finite energy solution to (4.1) on the open interval I with energy E and:

$$\sup_k \| \phi \|_{ED[I]} \ \leqslant \ \epsilon(E) \tag{4.61}$$

then one also has:

$$\| \phi \|_{S[I]} \ \leqslant \ F(E) \ . \tag{4.62}$$

Finally, such a solution $\phi(t)$ extends in a regular way to a neighborhood of the closure of the interval I.

For the remainder of this section we provide an outline of the proof of Theorem 4.7.

In order to construct the functions $F(E)$ and $\epsilon(E)$ such that (4.61) and (4.62) hold we use the induction on energy method. Precisely, we will show that there exists a strictly positive nonincreasing function defined for *all* values of E, $c_0 = c_0(E) \ll 1$, so that if the conclusion of the Theorem holds up to energy E then it also holds up to energy $E + c_0$. It is important here that c_0 depends *only* on E and not on the size of $F(E)$ or $\epsilon(E)$, as otherwise we would only be able to conclude the usual first step in an induction on energy proof, which is establishing that the set of regular energies is open.

According to Theorem 3.6 we know that $\epsilon(E)$ and $F(E)$ can be constructed up to some $E_0 \ll 1$. We now assume that E_0 is fixed by induction, and to increase its range we consider a solution ϕ defined on an interval I with energy $E[\phi] = E_0 + c$, $c \leqslant c_0(E_0)$ and with energy dispersion $\leqslant \epsilon$ (at first this is a free parameter which we may take as small as we like). We will compare ϕ with a wave map $\tilde{\phi}$ with energy E_0. To construct $\tilde{\phi}$ we reduce the initial data energy of $\phi[0]$ by truncation in frequency. We define the *cut frequency* $k_* \in \mathbb{R}$ according to (this can be done by adjusting the definition of the $P_{<k}$ continuously if necessary):

$$E[\Pi P_{\leqslant k_*}\phi[0]] \ = \ E_0 \ .$$

Here we work in the extrinsic setting, and the small energy dispersion insures that the low frequency projections $P_{\leq k}\phi_0$ stay close to the manifold. Then one can use any reasonable projection operator Π to return back to the manifold.

We consider the Wave-Map $\tilde{\phi}$ with this initial data $\tilde{\phi}[0] = \Pi P_{\leqslant k_*}\phi[0]$. This Wave-Map exists classically for at least a short amount of time according to Cauchy stability, and where it exists we have:

td_phi_eng
$$E[\tilde{\phi}(t)] \ = \ E_0 \ . \tag{4.63}$$

Since ϕ has energy dispersion $\leqslant \epsilon$, by (4.71) it follows that $\tilde{\phi}$ has energy dispersion $\lesssim_{E_0} \epsilon^{\frac{1}{4}}$ at time $t = 0$. Again by the usual Cauchy stability theory, if ϵ is chosen small enough in comparison to the inductively defined parameter $\epsilon(E_0)$ it follows that there exists a non-empty interval J_0 where $\tilde{\phi}$ satisfies:

en_disp_t
$$\sup_k \| P_k \tilde{\phi} \|_{L^\infty_t(L^\infty_x)[J_0]} \ \leqslant \ \epsilon(E_0) \ . \tag{4.64}$$

Then our induction hypothesis guarantees that we have the dispersive bounds:

hi_ind_bnd
$$\| \tilde{\phi} \|_{S[J_0]} \ \leqslant \ F(E_0) \ . \tag{4.65}$$

The plan is now very simple. On one hand, we try to pass the space-time control (i.e. the S bound) from $\tilde{\phi}$ to ϕ via linearization around $\tilde{\phi}$ to control the low frequencies, and conservation of energy and perturbation theory to control the high frequencies. On the other hand, we need to pass the good energy dispersion bounds from ϕ back down to $\tilde{\phi}$ in order to increase the size of $J \subseteq I$ on which (4.64) holds, until it eventually fills up all of I.

To summarize, we have the two wave maps $\tilde{\phi}$ and ϕ on an interval I with energies E, respectively $E + c$, so that

know-boot
$$\|\tilde{\phi}\|_S \leq \tilde{F} = F(E_0), \qquad \|\phi\|_{ED} \leq \epsilon \tag{4.66}$$

and we want to prove that

need-boot
$$\|\tilde{\phi}\|_{ED} \leq \tilde{\epsilon} = \epsilon(E_0), \qquad \|\phi\|_S \leq F \tag{4.67}$$

In doing this, we can freely make the bootstrap assumption

$$\|\tilde{\phi}\|_{ED} \leq 2\tilde{\epsilon}, \qquad \|\phi\|_S \leq 2F \tag{4.68}$$

We are also free to independently choose F sufficiently large and ϵ sufficiently small. But the delicate part is that c can only depend on E. The analysis is carried out in several steps:

4.6.1. *Energy dispersion and multilinear estimates.* Ideally, one would like to know that having small energy dispersion improves the multilinear bounds in Proposition 4.3. To understand this better let us first discuss the null form estimate (4.8) in the easiest case when both inputs are free waves. As discussed earlier, there there is an angular gain for small angle interactions, so one only needs to consider large angles, i.e. bilinear estimates for transversal waves. In that case the null form does not help, so we just treat this as a bilinear product estimate.

On one hand, using Strichartz estimates for one factor and the energy dispersion for the other we obtain an improved L^6 product estimate. On the other hand, the large angle bilinear estimate of Wolff [50] and Tao [44] shows that one also has an $L^{\frac{5}{3}}$ bound (the exact exponent does not matter, only that it is less than 2). Interpolating, one obtains an improved L^2 bound. That suffices, because the output of transversal free waves is at high modulation.

One downside of the above reasoning is that in the case of unbalanced frequency interactions one ends up with the wrong balance of the powers of the two frequencies, namely with an estimate of the type

$$\|\partial^\alpha \phi_j \partial_\alpha \phi_k\|_N \lesssim 2^{c|j-k|} \|\phi_j[0]\|_{\dot H^1\times L^2} \|\phi_k[0]\|^{1-\delta}_{\dot H^1\times L^2} \|\phi_k\|^{\delta}_{L^\infty}, \qquad c,\delta>0$$

Hence this energy dispersion gain is effective only in the case of balanced factors.

Ideally one would like to have the same estimate for S inputs. While this is not out of question, we were unable to prove that. Instead, we only have weaker estimates of the form

$$\|\partial^\alpha \phi_j \partial_\alpha \phi_k\|_N \lesssim 2^{c|j-k|} (\|\phi_j\|_{\dot H^1\times L^2} + \|\Box\phi_j\|_N)(\|\phi_k[0]\|_{\dot H^1\times L^2} + \|\Box \phi_k\|_N)^{1-\delta} \|\phi_k\|^{\delta}_{L^\infty}, \qquad c,\delta>0$$

The dyadic portions of our wave maps do not have this regularity. However, they do have it after renormalization. This is the reason why we introduce the following definition:

Definition 4.8. *(Renormalizable Functions) We define a non-linear functional $\mathcal{W}_k$ on S as follows:*

$$\begin{aligned} \| \phi \|_{\mathcal{W}_k} \ :=\ & \inf_{U\in SO(d)} \Big[\big(\| U \|_{S\cap \underline{X}} + \sup_{j\geqslant k} 2^{C(j-k)} \| P_j U \|_{S\cap \underline{X}} \big) \\ & \cdot\ \sup_{k'} 2^{|k'-k|} \big(\| P_{k'}(U\phi_k)[0] \|_{\dot H^1\times L^2} + \| P_{k'} \Box (U\phi_k) \|_N \big) \Big] \ . \end{aligned} \tag{4.69}$$

Using this notation, the above bound is improved to

$$\|\partial^\alpha \phi_j \partial_\alpha \phi_k\|_N \lesssim 2^{c|j-k|} \|\phi_j\|_{\mathcal{W}_j} \|\phi_k\|^{1-\delta}_{\mathcal{W}_k} \|\phi_k\|^{\delta}_{L^\infty}, \qquad c,\delta>0 \tag{4.70}$$

Similar improvements apply to the other multilinear estimates in Proposition 4.3, provided that at least two of the interacting frequencies are balanced.

These improved estimates are crucial in order to gain the large gap m which is needed in Proposition 4.6.

4.6.2. *Compare the initial data of ϕ and $\tilde\phi$.* At the linearized level we have $\tilde\phi[0] = P_{<k_*}\phi[0]$. This is not an identity, but the errors are higher order, and they will be small due to the energy dispersion,

$$\| P_k \big(P_{<k_*}\phi[0] - \tilde\phi[0] \big) \|_{\dot H^1\times L^2} \ \lesssim_E \ \epsilon^{\frac14} 2^{-\frac12 |k-k_*|} \ . \tag{4.71}$$

4.6.3. *Compare the low frequencies of ϕ and $\tilde{\phi}$.* The previous step shows that the low frequencies of the data for ϕ and $\tilde{\phi}$ are very close. Here we aim to show that a similar bound holds for the difference of the solutions,

cutphiS
$$\| P_k\big(P_{<k_*}\phi - \tilde{\phi}\big) \|_S \lesssim_F 2^{-\delta_0|k-k_*|}\epsilon^{\delta_0} \tag{4.72}$$

This yields the small energy dispersion for $\tilde{\phi}$, provided that ϵ is small enough. To prove (4.72) we consider the equation for the difference $\psi = P_{<k_*}\phi - \tilde{\phi}$. This has the form

$$\begin{aligned} \Box\psi = & -\mathcal{S}(\tilde{\phi})\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} + P_{<k_*}(\mathcal{S}(\phi)\partial^\alpha\phi\partial_\alpha\phi) \\ = & -\mathcal{S}(\tilde{\phi})\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} + \mathcal{S}(\tilde{\phi}+\psi)\partial^\alpha(\tilde{\phi}+\psi)\partial_\alpha(\tilde{\phi}+\psi) + R(\phi) \end{aligned}$$

where

$$R(\phi) = P_{<k_*}(\mathcal{S}(\phi)\partial^\alpha\phi\partial_\alpha\phi) - \mathcal{S}(P_{<k^*}\phi)\partial^\alpha P_{<k^*}\phi\partial_\alpha P_{<k^*}\phi$$

We rewrite the above equation in the paradifferential form

$$\Box\psi_k = -2\mathcal{A}^\alpha_{<k-m}(\tilde{\phi})\partial_\alpha\psi_k + Err_k(\psi) + P_k R(\phi)$$

Provided ϵ is small enough, the remaining part evolves essentially along the linearized flow along $\tilde{\phi}$, and can be solved perturbatively using the linear covariant estimates in Proposition 4.6, with respect to a norm defined as in (4.72). It remains to establish good estimates for the last two terms on the right.

The term $R(\phi)$ is estimated in N using the S norm for ϕ and its energy dispersion,

$$\|P_k R(\phi)\|_N \lesssim_F 2^{-\delta_0|k-k_*|}\epsilon^{\delta_0} \tag{4.73}$$

The term $Err_k(\psi)$ is at least quadratic in ψ. It is estimated directly, using Proposition 4.3 for unbalanced frequency interactions, and its energy dispersed improvement for the balanced ones. We remark that here we use the energy dispersion of $\tilde{\phi}$, but that is still can be assumed to be small enough to defeat the S norm of ψ.

4.6.4. *Compare the high frequencies.* Here we estimate directly the difference $\psi = \phi - \tilde{\phi}$,

high-diff
$$\|\phi - \tilde{\phi}\|_S \lesssim_{\tilde{F}} 1 \tag{4.74}$$

This yields the S bound for ϕ in (4.67). The tricky bit is to do this with a constant c which depends only on E and not on $\tilde{F}$.

The function ψ has initial data of size c, and solves the equation

$$\Box\psi = -\mathcal{S}(\tilde{\phi})\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} + \mathcal{S}(\tilde{\phi}+\psi)\partial^\alpha(\tilde{\phi}+\psi)\partial_\alpha(\tilde{\phi}+\psi)$$

We need to estimate only its high frequencies, i.e. larger than k_*. The idea is to reduce the problem again to a perturbation of the gauge covariant equation (4.48) but this time with coefficients depending on ϕ rather than $\tilde{\phi}$. The difficulty is that the size of $\tilde{\phi}$ is large, and this would force the needed smallness of c to depend on $\tilde{F}$ rather than on E. To remedy this we need several intermediate steps:

(i) Establish uniform energy bounds for ψ in the energy norm, which do not depend on F. This is done using the energy estimates for both ϕ and $\tilde{\phi}$, combined with the bound (4.72), which guarantees their almost orthogonality.

(ii) Prove a partial divisibility result for the S norm of $\tilde{\phi}$, as follows:

Lemma 4.9. *Let $\tilde{\phi}$ be a wave map with energy E and S norm $\tilde{F}$. Then there exists a collection of subintervals $I = \cup_{i=1}^{K} I_i$, such that $K = K(\tilde{F})$ depends only on $\tilde{F}$, and such that the following bound holds on each I_i:*

fung_bnd

$$\| \tilde{\phi} \|_{S[I_i]} \lesssim_E 1 \ . \tag{4.75}$$

(iii) Use the perturbative argument to estimate the S norm of ψ in each interval I_k. In each interval we do have the small energy dispersion for ϕ, but all other constants depend only on E; hence the smallness condition on c will also depend only on E, and so will the S bound on ψ on I_k. On the other hand the number of intervals and thus the global S bound for ψ will depend on $\tilde{F}$.

nergy_sect

4.7. **Energy and Morawetz estimates.** The study of the large data problem for wave maps relies on the finite speed of propagation property of the wave equation. Because of this and of the small data result, the following conclusion follows:

If a wave map blows up at a point, then its energy must concentrate toward the tip of the light cone originating at that point. Similarly, if scattering fails, then it fails inside a light cone.

Thus, in order to study both blow-up and scattering, it suffices to consider finite energy wave maps inside a light cone. In one case we are interested in what happens at the tip of the light cone, in the other we are interested in what happens inside the cone but toward infinity. We will see that the two problems are virtually identical. The main tools in the study of the energy distribution inside the cone are the energy and the Morawetz estimates. These are described in the sequel.

4.7.1. *Notations.* We consider the forward light cone

$$C = \{0 \leq t < \infty, \ r \leq t\}$$

and its subsets

$$C_{[t_0,t_1]} = \{t_0 \leq t \leq t_1, \ r \leq t\} \ .$$

The lateral boundary of $C_{[t_0,t_1]}$ is denoted by $\partial C_{[t_0,t_1]}$. The time sections of the cone are denoted by

$$S_{t_0} = \{t = t_0, \ |x| \leq t\} \ .$$

We also use the translated cones

$$C^{\delta} = \{\delta \leq t < \infty, \ r \leq t - \delta\}$$

as well as the corresponding notations $C^{\delta}_{[t_0,t_1]}$, $\partial C^{\delta}_{[t_0,t_1]}$ and $S^{\delta}_{t_0}$ for $t_0 > \delta$.

For some of the computations below it is convenient to use the null frame

$$L \ = \ \partial_t + \partial_r \ , \qquad \bar{L} \ = \ \partial_t - \partial_r \ , \qquad \not{\partial} \ = \ r^{-1}\partial_\theta \ .$$

4.7.2. *The energy-momentum tensor.* A systematic way to derive both the energy and the Morawetz estimates is by using the energy-momentum tensor:

EM_tensor
$$T_{\alpha\beta}[\Phi] \; = \; g_{ij}(\Phi)\big[\partial_\alpha\phi^i\partial_\beta\phi^j - \frac{1}{2}m_{\alpha\beta}\,\partial^\gamma\phi^i\partial_\gamma\phi^j\big] \ , \tag{4.76}$$

with a well chosen vector-field. Here $\Phi = (\phi^1, \dots, \phi^n)$ is a set of local coordinates on the target manifold $(\mathbb{M}, g)$ and $(m_{\alpha\beta})$ stands for the Minkowski metric. The main two properties of $T_{\alpha\beta}[\Phi]$ are:

- It is divergence free, $\nabla^\alpha T_{\alpha\beta} = 0$;
- It obeys the positive energy condition $T(X, Y) \geqslant 0$ whenever both $m(X, X) \leqslant 0$ and $m(Y, Y) \leqslant 0$.

Our estimates are obtained by contracting the energy-momentum tensor with a well chosen vector-field. The above properties imply that contracting $T_{\alpha\beta}[\Phi]$ with timelike/null vector-fields will result in good energy estimates on characteristic and space-like hypersurfaces.

If X is some vector-field, we can form its associated momentum density (i.e. its Noether current)

$$^{(X)}P_\alpha \; = \; T_{\alpha\beta}[\Phi]X^\beta \ .$$

This one form obeys the divergence rule

divergence
$$\nabla^{\alpha}\, {}^{(X)}P_\alpha \; = \; \frac{1}{2}T_{\alpha\beta}[\Phi]^{(X)}\pi^{\alpha\beta} \ , \tag{4.77}$$

where $^{(X)}\pi_{\alpha\beta}$ is the deformation tensor of X,

$$^{(X)}\pi_{\alpha\beta} \; = \; \nabla_\alpha X_\beta + \nabla_\beta X_\alpha \ .$$

A simple computation shows that one can also express

$$^{(X)}\pi \; = \; \mathcal{L}_X g \ .$$

This latter formulation is very convenient when dealing with coordinate derivatives. Recall that in general one has:

$$(\mathcal{L}_X g)_{\alpha\beta} \; = \; X(g_{\alpha\beta}) + \partial_\alpha(X^\gamma)g_{\gamma\beta} + \partial_\beta(X^\gamma)g_{\alpha\gamma} \ .$$

The energy estimates are obtained by integrating the relation (4.77) over cones $C^\delta_{[t_1,t_2]}$. Then from (4.77) we obtain, for $\delta \leq t_1 \leq t_2$:

enest
$$\int_{S^\delta_{t_2}} {}^{(X)}P_0 \; dx + \frac{1}{2}\int_{C^\delta_{[t_1,t_2]}} T_{\alpha\beta}[\Phi]^{(X)}\pi^{\alpha\beta} \; dxdt = \int_{S^\delta_{t_1}} {}^{(X)}P_0 \; dx + \int_{\partial C^\delta_{[t_1,t_2]}} {}^{(X)}P_L \; dA \ , \tag{4.78}$$

where dA is an appropriately normalized (Euclidean) surface area element on the lateral boundary of the cone $r = t - \delta$.

4.7.3. *Energy estimates.* The standard energy estimates come from contracting $T_{\alpha\beta}[\Phi]$ with $Y = \partial_t$. Then we have

$$^{(Y)}\pi = 0 \ , \qquad {}^{(Y)}P_0 = \frac{1}{2}(|\partial_t\Phi|^2 + |\nabla_x\Phi|^2) \ , \qquad {}^{(Y)}P_L = \frac{1}{4}|L\Phi|^2 + \frac{1}{2}|\not{\partial}\Phi|^2 \ .$$

Applying (4.78) over $C_{[t_1,t_2]}$ we obtain the energy-flux relation

en-flux
$$E_{S_{t_1}}[\Phi] = E_{S_{t_0}}[\Phi] + F_{[t_0,t_1]}[\Phi] \ . \tag{4.79}$$

where E_{S_t} represents the energy of Φ on time sections,

$$E_{S_t}[\Phi] = \frac{1}{2}\int_{S_t} (|\partial_t \Phi|^2 + |\nabla_x \Phi|^2) dx \ .$$

and $F_{[t_0,t_1]}[\Phi]$ represents the lateral flux of Φ between t_0 and t_1 as

$$F_{[t_0,t_1]}[\Phi] = \int_{\partial C_{[t_0,t_1]}} \big(\frac{1}{4}|L\Phi|^2 + \frac{1}{2}|\not{\partial}\Phi|^2\big) dA \ .$$

The energy relation (4.79) shows that $E_{S_t}[\Phi]$ is a nondecreasing function of t. It also shows that for the blow-up problem we have

$$\lim_{t_1,t_2\to 0} F_{[t_1,t_2]}[\Phi] = 0$$

and for the scattering problem we have

$$\lim_{t_1,t_2\to \infty} F_{[t_1,t_2]}[\Phi] = 0 \ .$$

This is the main decay estimate arising as a consequence of the energy relation. Later we will want to turn the flux decay on the boundary of the cone into an integrated decay inside the cone. This is accomplished using Morawetz type estimates.

Finally, we remark that applying (4.78) over $C^\delta_{[\delta,1]}$ yields

enyd
$$\int_{\partial C^\delta_{[\delta,1]}} \frac{1}{4}|L\Phi|^2 + \frac{1}{2}|\not{\partial}\Phi|^2 \ dA \leq E_1[\Phi] \ . \tag{4.80}$$

This will be used later on.

4.7.4. *The energy of self-similar maps.* A map $\Phi : C \to (M, g)$ is self-similar if

$$\Phi(\lambda t, \lambda x) = \Phi(t, x), \qquad (t, x) \in C, \quad \lambda > 0$$

Such a map, if it had finite energy, would be a natural obstruction to global existence of wave maps. Later we will argue that finite energy self-similar wave maps do not exist. Here we carry out a preliminary step, which is to compute the energy $E[\Phi]$ (which is independent of time) in hyperbolic coordinates.

Hyperbolic coordinates (ρ, y, Θ) are introduced inside C via

hyp_coords
$$t \ = \ \rho\cosh(y) \ , \qquad\qquad r \ = \ \rho\sinh(y) \ , \qquad\qquad \theta \ = \ \Theta \ , \tag{4.81}$$

and self-similar maps Φ can be viewed as functions $\Phi = \Phi(y, \Theta)$ on $\mathbb{H}^2$.

In this system of coordinates, the Minkowski metric becomes

metric
$$-dt^2 + dr^2 + r^2 d\theta^2 \ = \ -d\rho^2 + \rho^2\big(dy^2 + \sinh^2(y) d\Theta^2\big) \ . \tag{4.82}$$

A quick calculation shows that the contraction on line (4.84) becomes the one-form

$$^{(Y)}P^\alpha dV_\alpha \ = \ T(\partial_\rho, \partial_t)\rho^2 dA_{\mathbb{H}^2} \ , \qquad dA_{\mathbb{H}^2} \ = \ \sinh(y) dy d\Theta \ . \tag{4.83}$$

The area element $dA_{\mathbb{H}^2}$ is that of the hyperbolic plane $\mathbb{H}^2$. To continue, we note that:

$$\partial_t \ = \ \frac{t}{\rho}\partial_\rho - \frac{r}{\rho^2}\partial_y \ ,$$

so in particular

$$T(\partial_\rho, \partial_t) \;=\; \frac{\cosh(y)}{2}|\partial_\rho\Phi|^2 - \frac{\sinh(y)}{\rho}\partial_\rho\Phi\cdot\partial_y\Phi + \frac{\cosh(y)}{2\rho^2}\Big(|\partial_y\Phi|^2 + \frac{1}{\sinh^2(y)}|\partial_\Theta\Phi|^2\Big) \ .$$

This computation allows us to obtain a version of the usual energy estimate adapted to the hyperboloids $\sqrt{t^2-r^2}=1$. Integrating the divergence of the ${}^{(Y)}\!P_\alpha$ momentum density over regions of the form $\mathcal{R}=\{\rho\geqslant\rho_0, t\leqslant t_0\}$ we have:

hyp_eng

$$\int_{\{\rho=1\}\cap\{t\leqslant t_0\}} {}^{(Y)}\!P^\alpha dV_\alpha = \int_{\{\rho>1\}\cap\{t=t_0\}} {}^{(Y)}\!P_0 dx \tag{4.84}$$

where the integrand on the LHS denotes the interior product of ${}^{(Y)}\!P$ with the Minkowski volume element.

Letting $t_0\to\infty$ in (4.84) we obtain a useful consequence of this, namely a weighted hyperbolic space estimate for special solutions to the wave-map equations, which will be used in the sequel to rule out the existence of non-trivial finite energy self-similar solutions:

yp_eng_lem

Lemma 4.10. *Let Φ be a self-similar finite energy smooth wave-map in the interior of the cone C Then one has:*

o0_hyp_eng

$$\mathcal{E}[\Phi] = \frac{1}{2}\int_{\mathbb{H}^2} |\nabla_{\mathbb{H}^2}\Phi|^2\cosh(y) dA_{\mathbb{H}^2} \ . \tag{4.85}$$

Here:

$$|\nabla_{\mathbb{H}^2}\Phi|^2 \;=\; |\partial_y\Phi|^2 + \frac{1}{\sinh^2(y)}|\partial_\Theta\Phi|^2 \ ,$$

is the covariant energy density for the hyperbolic metric.

4.7.5. *Morawetz estimates.* Our goal here is to obtain decay estimates for time-like components of the energy density. For this we use the energy momentum estimate (4.78) with respect to the timelike/null vector-field

X_field

$$X_\epsilon \;=\; \frac{1}{\rho_\epsilon}\big((t+\epsilon)\partial_t + r\partial_r\big)\,, \qquad \rho_\epsilon = \sqrt{(t+\epsilon)^2 - r^2} \ . \tag{4.86}$$

In order to gain some intuition, we first consider the case of X_0. This is most readily expressed in the system of hyperbolic coordinates (4.81). One easily checks that the coordinate derivatives turn out to be

$$\partial_\rho \;=\; X_0 \ , \qquad\qquad \partial_y \;=\; r\partial_t + t\partial_r \ .$$

In particular, X_0 is uniformly timelike with $m(X_0,X_0)=-1$, and one should expect it to generate good energy estimate on time slices $t=const.$ In the system of coordinates (4.81) one also has that

$$\mathcal{L}_{X_0} m \;=\; 2\rho\big(dy^2 + \sinh^2(y)d\Theta^2\big) \ .$$

Raising indices, one then computes

$${}^{(X_0)}\!\pi^{\alpha\beta} \;=\; \frac{2}{\rho^3}\big(\partial_y\otimes\partial_y + \sinh^{-2}(y)\partial_\Theta\otimes\partial_\Theta\big) \ .$$

Therefore, we have the contraction identity:

$$\frac{1}{2}T_{\alpha\beta}[\Phi]^{(X_0)}\!\pi^{\alpha\beta} \;=\; \frac{1}{\rho}|X_0\Phi|^2 \ .$$

To compute the components of ${}^{(X_0)}\!P_0$ and ${}^{(X_0)}\!P_L$ we use the associated optical functions

$$u \ = \ t - r \ , \qquad v \ = \ t + r \ , \qquad uv = \rho^2$$

Then we have

$$X_0 \ = \ \frac{1}{\rho}\left(\frac{1}{2}vL + \frac{1}{2}u\bar{L}\right) \ , \qquad\qquad \partial_t \ = \ \frac{1}{2}L + \frac{1}{2}\bar{L} \ . \tag{4.87}$$

Finally, we record here the components of $T_{\alpha\beta}[\Phi]$ in the null frame

$$T(L,L) \ = \ |L\Phi|^2 \ , \qquad T(\bar{L},\bar{L}) \ = \ |\bar{L}\Phi|^2 \ , \qquad T(L,\bar{L}) \ = \ |\not{\partial}\Phi|^2 \ .$$

By combining the above calculations, we see that we may compute

$$\begin{aligned} {}^{(X_0)}\!P_0 = T(\partial_t, X_0) &= \frac{1}{4}\left(\frac{v}{u}\right)^{\frac{1}{2}}|L\Phi|^2 + \frac{1}{4}\Big[\left(\frac{v}{u}\right)^{\frac{1}{2}} + \left(\frac{u}{v}\right)^{\frac{1}{2}}\Big]|\not{\partial}\Phi|^2 + \frac{1}{4}\left(\frac{u}{v}\right)^{\frac{1}{2}}|\bar{L}\Phi|^2 \ , \\ {}^{(X_0)}\!P_L = T(L, X_0) &= \frac{1}{2}\left(\frac{v}{u}\right)^{\frac{1}{2}}|L\Phi|^2 + \frac{1}{2}\left(\frac{u}{v}\right)^{\frac{1}{2}}|\not{\partial}\Phi|^2 \ . \end{aligned}$$

These are essentially the same as the components of the usual energy currents ${}^{(\partial_t)}\!P_0$ and ${}^{(\partial_t)}\!P_L$ modulo ratios of the optical functions u and v.

One would expect to get nice space-time estimates for $X_0\Phi$ by integrating (4.77) over the interior cone $r \leqslant t \leqslant 1$. The problem is that the boundary terms degenerate when $\rho \to 0$. To avoid this difficulty we simply redo everything with the shifted version X_ϵ from line (4.86). The above formulas remain valid with u, v replaced by their time shifted versions

$$u_\epsilon \ = \ (t+\epsilon) - r \ , \qquad\qquad v_\epsilon \ = \ (t+\epsilon) + r \ .$$

Furthermore, notice that for small t one has the bounds

$$\left(\frac{v_\epsilon}{u_\epsilon}\right)^{\frac{1}{2}} \approx 1 \ , \qquad \left(\frac{u_\epsilon}{v_\epsilon}\right)^{\frac{1}{2}} \approx 1 \ , \qquad 0 < t \leq \epsilon$$

within the cone C. Thus,

$${}^{(X_\epsilon)}\!P_0 \ \approx \ {}^{(\partial_t)}\!P_0 \ , \qquad 0 < t \leq \epsilon \ .$$

In what follows we work with a wave-map Φ in $C_{[\epsilon,1]}$. We denote its total energy and flux by

$$E = E_{S_1}[\Phi] \ , \qquad F = F_{[\epsilon,1]}[\Phi] \ .$$

In the limiting case $F = 0$, $\epsilon = 0$ one could apply (4.78) to obtain

$$\int_{S^0_{t_2}} {}^{(X_\epsilon)}\!P_0 \ dx + \int_{C^0_{[t_1,t_2]}} \frac{1}{\rho_\epsilon}|X_\epsilon\Phi|^2 \ dxdt = \int_{S^0_{t_1}} {}^{(X_\epsilon)}\!P_0 \ dx \ .$$

By (4.88), letting $t_1 \to 0$ followed by $\epsilon \to 0$ and taking supremum over $0 < t_2 \leqslant 1$ we would get the model estimate

$$\sup_{t\in(0,1]} \int_{S^0_t} {}^{(X_0)}\!P_0 \ dx + \int_{C^0_{[0,1]}} \frac{1}{\rho}|X_0\Phi|^2 \ dxdt \leq E \ .$$

However, here we need to deal with a small nonzero flux. Observing that

$${}^{(X_\epsilon)}\!P_L \ \lesssim \ \epsilon^{-\frac{1}{2}}\, {}^{(\partial_t)}\!P_L \ ,$$

from (4.78) we obtain the weaker bound

$$\int_{S^0_{t_2}} {}^{(X_\epsilon)}\!P_0 \ dx + \int_{C^0_{[t_1,t_2]}} \frac{1}{\rho_\epsilon}|X_\epsilon\Phi|^2 \ dxdt \lesssim \int_{S^0_{t_1}} {}^{(X_\epsilon)}\!P_0 \ dx + \epsilon^{-\frac{1}{2}}F \ .$$

Letting $t_1 = \epsilon$ and taking supremum over $\epsilon \leqslant t_2 \leqslant 1$ we obtain

enxe
$$\sup_{t\in(\epsilon,1]} \int_{S^0_t} {}^{(X_\epsilon)}\!P_0 \ dx + \int_{C^0_{[0,1]}} \frac{1}{\rho_\epsilon}|X_\epsilon\Phi|^2 \ dxdt \lesssim E + \epsilon^{-\frac{1}{2}}F \tag{4.88}$$

A consequence of this is the following, which will be used to rule out the case of asymptotically null pockets of energy:

Lemma 4.11. *Let Φ be a smooth wave-map in the cone $C_{(\epsilon,1]}$ which satisfies the flux-energy relation $F \lesssim \epsilon^{\frac{1}{2}}E$. Then*

enxea
$$\int_{S^0_1} {}^{(X_\epsilon)}\!P_0 \ dx \lesssim E \ . \tag{4.89}$$

Next, we show can replace X_ϵ by X_0 in (4.88) if we restrict the integrals on the left to $r < t - \epsilon$. In this region we have

$${}^{(X_\epsilon)}\!P_0 \approx {}^{(X_0)}\!P_0 \ , \qquad \rho_\epsilon \approx \rho \ .$$

In addition, a direct computation shows that in $r < t - \epsilon$

$$\frac{1}{\rho}|X_0\Phi|^2 \lesssim \frac{1}{\rho_\epsilon}|X_\epsilon\Phi|^2 + \frac{\epsilon^2}{\rho^3}|\partial_t\Phi|^2$$

and also

$$\int_{C^\epsilon_{(\epsilon,1]}} \frac{\epsilon^2}{\rho^3}|\partial_t\Phi|^2 dxdt \leq \int_{C^\epsilon_{(\epsilon,1]}} \frac{\epsilon^{\frac{1}{2}}}{t^{\frac{3}{2}}}|\partial_t\Phi|^2 dxdt \lesssim E \ .$$

Thus, using the last three relations in (4.88) we have proved the following estimate which will be used to conclude that rescaling of Φ are asymptotically stationary, and also used to help trap uniformly time-like pockets of energy:

Lemma 4.12. *Let Φ be a smooth wave-map in the cone $C_{(\epsilon,1]}$ which satisfies the flux-energy relation $F \lesssim \epsilon^{\frac{1}{2}}E$. Then we have*

xphi
$$\sup_{t\in(\epsilon,1]} \int_{S^\epsilon_t} {}^{(X_0)}\!P_0 \ dx + \int_{C^\epsilon_{[\epsilon,1]}} \frac{1}{\rho}|X_0\Phi|^2 dxdt \lesssim E \ . \tag{4.90}$$

Finally, we use the last lemma to propagate pockets of energy forward away from the boundary of the cone. By (4.78) for X_0 we have

$$\int_{S^\delta_1} {}^{(X_0)}\!P_0 \ dx \leq \int_{S^\delta_{t_0}} {}^{(X_0)}\!P_0 \ dx + \int_{\partial C^\delta_{[t_0,1]}} {}^{(X_0)}\!P_L \ dA \ , \qquad \epsilon \leq \delta < t_0 < 1 \ .$$

We consider the two components of ${}^{(X_0)}\!P_L$ separately. For the angular component, by (4.80) we have the bound

$$\int_{\partial C^\delta_{[t_0,1]}} \left(\frac{u}{v}\right)^{\frac{1}{2}} |\partial\!\!\!/\Phi|^2 \ dA \lesssim \left(\frac{\delta}{t_0}\right)^{\frac{1}{2}} \int_{\partial C^\delta_{[t_0,1]}} |\partial\!\!\!/\Phi|^2 \ dA \lesssim \left(\frac{\delta}{t_0}\right)^{\frac{1}{2}} E \ .$$

For the L component a direct computation shows that

$$|L\Phi| \lesssim \left(\frac{u}{v}\right)^{\frac{1}{2}} |X_0\Phi| + \left(\frac{u}{v}\right) |\bar{L}\Phi| \ .$$

Thus we obtain

$$\int_{S_1^\delta} {}^{(X_0)}\!P_0 \ dx \lesssim \int_{S_{t_0}^\delta} {}^{(X_0)}\!P_0 \ dx + \left(\frac{\delta}{t_0}\right)^{\frac{1}{2}} E + \int_{\partial C^\delta_{[t_0,1]}} \Big(\left(\frac{u}{v}\right)^{\frac{1}{2}} |X_0\phi|^2 + \left(\frac{u}{v}\right)^{\frac{3}{2}} |\bar{L}\Phi|^2 \Big) dA \ .$$

For the last term we optimize with respect to $\delta \in [\delta_0, \delta_1]$ to obtain:

Lemma 4.13. *Let Φ be a smooth wave-map in the cone $C_{(\epsilon,1]}$ which satisfies the flux-energy relation $F \lesssim \epsilon^{\frac{1}{2}} E$. Suppose that $\epsilon \leq \delta_0 \ll \delta_1 \leq t_0$. Then*

$$\int_{S_1^{\delta_1}} {}^{(X_0)}\!P_0 \ dx \lesssim \int_{S_{t_0}^{\delta_0}} {}^{(X_0)}\!P_0 \ dx + \left(\left(\frac{\delta_1}{t_0}\right)^{\frac{1}{2}} + (\ln(\delta_1/\delta_0))^{-1} \right) E \ . \tag{4.91}$$

To prove this lemma, it suffices to choose $\delta \in [\delta_0, \delta_1]$ so that

$$\int_{\partial C^\delta_{[t_0,1]}} \left[\left(\frac{u}{v}\right)^{\frac{1}{2}} |X_0\phi|^2 + \left(\frac{u}{v}\right)^{\frac{3}{2}} |\bar{L}\Phi|^2 \right] \ dA \lesssim |\ln(\delta_1/\delta_0)|^{-1} E \ .$$

This follows by pigeonholing the estimate

$$\int_{C^{\delta_0}_{[t_0,1]} \setminus C^{\delta_1}_{[t_0,1]}} \frac{1}{u} \left[\left(\frac{u}{v}\right)^{\frac{1}{2}} |X_0\phi|^2 + \left(\frac{u}{v}\right)^{\frac{3}{2}} |\bar{L}\Phi|^2 \right] dx dt \lesssim E \ .$$

The first term is estimated directly by (4.90). For the second we simply use energy bounds since in the domain of integration we have the relation

$$\frac{1}{u}\left(\frac{u}{v}\right)^{\frac{3}{2}} \leq \frac{\delta_1^{\frac{1}{2}}}{t^{\frac{3}{2}}} \ .$$

4.8. **The threshold theorem.** Using the energy dispersed result in Theorem 4.7 and the energy/Morawetz estimates in the previous section we can now approach the large data problem. For the blow-up question we prove the following:

[38] *Let $\Phi : C_{(0,1]} \to \mathbb{M}$ be a C^∞ wave map. Then exactly one of the following possibilities must hold:*

Theorem 4.14. (1) *There exists a sequence of points $(t_n, x_n) \in C_{[0,1]}$ and scales r_n with*

$$(t_n, x_n) \to (0,0) \ , \qquad \limsup \frac{|x_n|}{t_n} < 1 \ , \qquad \lim \frac{r_n}{t_n} = 0$$

so that the rescaled sequence of wave-maps

$$\Phi^{(n)}(t,x) \ = \ \Phi\big(t_n + r_n t, x_n + r_n x\big) \ , \tag{4.92}$$

converges strongly in H^1_{loc} to a Lorentz transform of an entire Harmonic-Map of nontrivial energy:

$$\Phi^{(\infty)} : \mathbb{R}^2 \to \mathbb{M} \ , \qquad 0 < \| \, \Phi^{(\infty)} \, \|_{\dot{H}^1(\mathbb{R}^2)} \leqslant \lim_{t \to 0} E_{S_t}[\Phi] \ .$$

(2) *For each $\epsilon > 0$ there exists $0 < t_0 \leqslant 1$ and a wave map extension*

$$\Phi : \mathbb{R}^2 \times (0, t_0] \to \mathbb{M}$$

with bounded energy

caseben
$$E[\Phi] \leq (1 + \epsilon^8) \lim_{t \to 0} E_{S_t}[\Phi] \tag{4.93}$$

and energy dispersion,

casebed
$$\sup_{t \in (0, t_0]} \sup_{k \in \mathbb{Z}} \big(\| P_k \Phi(t) \|_{L^\infty_x} + 2^{-k} \| P_k \partial_t \Phi(t) \|_{L^\infty_x} \big) \leq \epsilon \ . \tag{4.94}$$

maint

The analogue result for the scattering problem also holds:

Theorem 4.15. *Let $\Phi : C_{[1,\infty)} \to \mathbb{M}$ be a C^∞ wave map with finite energy. Then exactly one of the following possibilities must hold:*

(1) *There exists a sequence of points $(t_n, x_n) \in C_{[1,\infty)}$ and scales r_n with*

$$t_n \to \infty \ , \qquad \limsup \frac{|x_n|}{t_n} < 1 \ , \qquad \lim \frac{r_n}{t_n} = 0$$

so that the rescaled sequence of wave-maps

rescalings
$$\Phi^{(n)}(t, x) \ = \ \Phi\big(t_n + r_n t, x_n + r_n x\big) \ , \tag{4.95}$$

converges strongly in H^1_{loc} to a Lorentz transform of an entire Harmonic-Map of nontrivial energy:

$$\Phi^{(\infty)} : \mathbb{R}^2 \to \mathbb{M} \ , \qquad 0 < \| \, \Phi^{(\infty)} \, \|_{\dot{H}^1(\mathbb{R}^2)} \leqslant \lim_{t \to \infty} E_{S_t}[\Phi] \ .$$

(2) *For each $\epsilon > 0$ there exists $t_0 > 1$ and a wave map extension*

$$\Phi : \mathbb{R}^2 \times [t_0, \infty) \to \mathbb{M}$$

with bounded energy

casebens
$$E[\Phi] \leq (1 + \epsilon^8) \lim_{t \to \infty} E_{S_t}[\Phi] \tag{4.96}$$

and energy dispersion,

casebeds
$$\sup_{t \in [t_0, \infty)} \sup_{k \in \mathbb{Z}} \big(\| P_k \Phi(t) \|_{L^\infty_x} + 2^{-k} \| P_k \partial_t \Phi(t) \|_{L^\infty_x} \big) \leq \epsilon \ . \tag{4.97}$$

maints

We recall that a nontrivial harmonic map $\Phi^{(\infty)} : \mathbb{R}^2 \to \mathbb{M}$ cannot have an arbitrarily small energy. Precisely, there are two possibilities. Either there are no such harmonic maps (for instance, in the case when $\mathbb{M}$ is negatively curved, see [24]) or there exists a lowest energy nontrivial harmonic map, which we have denoted by $E_{crit} > 0$. Furthermore, a simple computation shows that the energy of any harmonic map will increase if we apply a Lorentz transformation. Hence, combining the results of Theorem 4.14 and Theorem 4.60 we obtain the following:

Corollary 4.16. *(Global Regularity for Wave-Maps) The following statements hold:*

(1) *Assume that* $\mathbb{M}$ *is a compact Riemannian manifold so that there are no nontrivial finite energy harmonic maps* $\Phi^{(\infty)} : \mathbb{R}^2 \to \mathbb{M}$. *Then for any finite energy data* $\Phi[0] : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{M} \times T\mathbb{M}$ *for the wave map equation there exists a global solution* $\Phi \in S$. *In addition, this global solution retains any additional regularity of the initial data.*

(2) *Let* $\pi : \tilde{\mathbb{M}} \to \mathbb{M}$ *be a Riemannian covering, with* $\mathbb{M}$ *compact, and such that there are no nontrivial finite energy harmonic maps* $\Phi^{(\infty)} : \mathbb{R}^2 \to \mathbb{M}$. *If* $\Phi[0] : \mathbb{R}^2 \times \mathbb{R}^2 \to \tilde{\mathbb{M}} \times T\tilde{\mathbb{M}}$ *is* C^∞, *then there is a global* C^∞ *solution to* $\tilde{\mathbb{M}}$ *with this data.*

(3) *Suppose that there exists a lowest energy nontrivial harmonic map into* $\mathbb{M}$ *with energy* E_{crit}. *Then for any data* $\Phi[0] : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{M} \times T\mathbb{M}$ *for the wave map equation with energy below* E_{crit}, *there exists a global solution* $\Phi \in S$.

tccor

We remark that the statement in part (2) is a simple consequence of (1) and restricting the projection $\pi \circ \Phi$ to a sufficiently small section S_t of a cone where one expects blowup of the original map into $\tilde{\mathbb{M}}$. In particular, since this projection is regular by part A), its image lies in a simply connected set for sufficiently small t. Thus, this projection can be inverted to yield regularity of the original map close to the suspected blowup point. Because of this trivial reduction, we work exclusively with compact $\mathbb{M}$ in the sequel. It should be remarked however, that as a (very) special case of this result one obtains global regularity for smooth Wave-Maps into all hyperbolic spaces $\mathbb{H}^n$, which has been a long-standing and important conjecture in geometric wave equations due to its relation with problems in general relativity (see Chapter 16 of [11]).

The statement of Corollary 4.16 in its full generality was known as the *Threshold Conjecture*. Similar results were previously established for the Wave-Map problem via symmetry reductions in the works [12], [35], [41], and [40].

The proof of Theorems 4.14, 4.15 are similar, and are outlined in what follows.

Step 1: Extension. Here one constructs an extension for small t in the blow-up problem, respectively for large t in the scattering problem, so that the energy is increased very little, as in (4.93), respectively (4.96).

This argument uses the flux decay in an essential way; this allows us to initiate the extension at a time t_0 where $\not\nabla\Phi$ is very small on the boundary ∂S_{t_0} of the cone, thus guaranteeing the smallness of the energy outside the cone.

By energy estimates this guarantees that the energy remains small outside the cone up to time zero for the blow-up problem, respectively up to time infinity for the scattering problem. By the small data result, this suffices in order to insure that our extended solution remains regular outside the cone.

Step 2: Energy dispersion and scaling. Here we work with the extensions constructed above. Either they have small energy dispersion, in which case we are done by the energy dispersion result in Theorem 4.7, or not, in which case we have a sequence of points (t_n, x_n) and frequencies k_n with either $t_n \to 0$ or $t_n \to \infty$, so that

$$|P_{k_n}\phi(t_n, x_n)| + 2^{-k_n}|P_{k_n}\partial_t\phi(t_n, x_n)| \geq \epsilon$$

Using also the flux decay in (4.79) and rescaling t_n to 1 we arrive at a setting where we have the sequence of wave maps

$$\Phi^{(n)}(t,x) = \Phi(t_n t, t_n x)$$

in the increasing regions $C_{[\epsilon_n,1]}$, with $\epsilon_n \to 0$, so that

$$F_{[\epsilon_n,1]}[\Phi^{(n)}] \leq \epsilon_n^{\frac{1}{2}} E[\Phi] \ ,$$

and also points $x_n \in \mathbb{R}^2$ and frequencies $k_n \in \mathbb{Z}$ so that

$$|P_{k_n}\Phi^{(n)}(1,x_n)| + 2^{-k_n}|P_{k_n}\partial_t\Phi^{(n)}(1,x_n)| > \epsilon \ . \tag{4.98}$$

From this point on, the proofs of Theorems 4.14 and 4.15 are identical.

Step 3: Elimination of null concentration scenario. Using the fixed time portion of the X_0 energy bounds in (4.89) we eliminate the case of null concentration

$$|x_n| \to 1 \ , \qquad k_n \to \infty$$

in estimate (4.98), and show that the sequence of maps $\Phi^{(n)}$ at time $t=1$ must either have low frequency concentration in the range:

$$m(\epsilon,E) < k_n < M(\epsilon,E) \ , \qquad |x_n| < R(\epsilon,E)$$

or high frequency concentration strictly inside the cone:

$$k_n \geq M(\epsilon,E), \qquad |x_n| < \gamma(\epsilon,E) < 1 \ .$$

Step 4: Time-like energy concentration. In both remaining cases above we show that a nontrivial portion of the energy of $\Phi^{(n)}$ at time 1 must be located inside a smaller cone,

$$\frac{1}{2}\int_{t=1,|x|<\gamma_1} \left(|\partial_t\Phi^{(n)}|^2 + |\nabla_x\Phi^{(n)}|^2\right) dx \geq E_1$$

where $E_1 = E_1(\epsilon,E)$ and $\gamma_1 = \gamma_1(\epsilon,E) < 1$.

Step 5: Uniform propagation of non-trivial time-like energy. Using again the X_0 energy bounds as in Lemma 4.13 we propagate the above time-like energy concentration for $\Phi^{(n)}$ from time 1 to smaller times $t \in [\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}]$,

$$\frac{1}{2}\int_{|x|<\gamma_2(\epsilon,E)t} \left(|\partial_t\Phi^{(n)}|^2 + |\nabla_x\Phi^{(n)}|^2\right) dx \geq E_0(\epsilon,E) \ , \qquad t \in [\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}] \ .$$

At the same time, we obtain bounds for $X_0\Phi^{(n)}$ outside smaller and smaller neighborhoods of the cone, namely

$$\int_{C^{\epsilon_n}_{[\epsilon_n^{\frac{1}{2}},\epsilon_n^{\frac{1}{4}}]}} \rho^{-1}|X_0\Phi^{(n)}|^2 dx dt \lesssim 1 \ .$$

Step 6: Final rescaling. By a pigeonhole argument and rescaling we end up producing another sequence of maps, denoted still by $\Phi^{(n)}$, which are sections the original wave map Φ and are defined in increasing regions $C_{[1,T_n]}$, $T_n = e^{|\ln\epsilon_n|^{\frac{1}{2}}}$, and satisfy the following three properties:

$$E_{S_t}[\Phi^{(n)}] \approx E, \quad t \in [1, T_n] \qquad \text{(Bounded Energy)}$$

$$E_{S_t^{(1-\gamma_2)t}}[\Phi^{(n)}] \geq E_2, \quad t \in [1, T_n] \qquad \text{(Nontrivial Time-like Energy)}$$

$$\int_{C^{\epsilon_n^{\frac{1}{2}}}_{[1,T_n]}} \frac{1}{\rho} |X_0 \Phi^{(n)}|^2 dx dt \lesssim |\log \epsilon_n|^{-\frac{1}{2}} \qquad \text{(Decay to Self-similar Mode)}$$

Step 7: Isolating the concentration scales. Using several additional pigeonholing arguments we show that one of the following two scenarios must occur:

(1) (Energy Concentration) On a subsequence there exist $(t_n, x_n) \to (t_0, x_0)$, with (t_0, x_0) inside $C^{\frac{1}{2}}_{[\frac{1}{2},\infty)}$, and scales $r_n \to 0$ so that we have

$$E_{B(x_n,r_n)}[\Phi^{(n)}](t_n) = \frac{1}{10} E_0 \ ,$$
$$E_{B(x,r_n)}[\Phi^{(n)}](t_n) \leq \frac{1}{10} E_0 \ , \qquad x \in B(x_0, r) \ ,$$
$$r_n^{-1} \int_{t_n - r_n/2}^{t_n + r_n/2} \int_{B(x_0,r)} |X_0 \Phi^{(n)}|^2 dx dt \to 0 \ .$$

(2) (Non-concentration) For each $j \in \mathbb{N}$ there exists an $r_j > 0$ such that for every (t, x) inside $C_j = C^1_{[1,\infty)} \cap \{2^j < t < 2^{j+1}\}$ one has

$$E_{B(x,r_j)}[\Phi^{(n)}](t) \leq \frac{1}{10} E_0 \ , \qquad \forall (t,x) \in C_j \ ,$$
$$E_{S_t^{(1-\gamma_2)t}}[\Phi^{(n)}](t) \geq E_2 \ ,$$
$$\int_{C_j} |X_0 \Phi^{(n)}|^2 dx dt \to 0 \ .$$

uniformly in n.

Here E_0 represents the threshold in the small data result.

Step 8: The compactness argument In case i) above we consider the rescaled wave-maps

$$\Psi^{(n)}(t, x) = \Phi^{(n)}(t_n + r_n t, x_n + r_n x)$$

and show that on a subsequence they converge locally in the energy norm to a finite energy nontrivial wave map Ψ in $\mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}]$ which satisfies $X(t_0, x_0)\Psi = 0$. Thus Ψ must be a Lorentz transform of a nontrivial harmonic map.

In case ii) above we show directly that the sequence $\Phi^{(n)}$ converges locally on a subsequence in the energy norm to finite energy nontrivial wave map Ψ, defined in the interior of a translated cone $C^2_{[2,\infty)}$, which satisfies $X_0\Psi = 0$. Consequently, in hyperbolic coordinates we may interpret Ψ as a nontrivial harmonic map

$$\Psi : \mathbb{H}^2 \to M \ .$$

Compactifying this and using conformal invariance, we obtain a non-trivial finite energy harmonic map

$$\Psi : \mathbb{D}^2 \to M$$

from the unit disk $\mathbb{D}^2$, which according to the estimates of Section 4.7 obeys the additional weighted energy bound:

$$\int_{\mathbb{D}^2} |\nabla_x \Phi|^2 \frac{dx}{1-r} \; < \; \infty \; .$$

But such maps do not exist due to a combination of a theorem of Qing [29] and a theorem of Lemaire [24].

4.9. **Further developments.** We begin with some comments concerning the higher dimensional case. First of all, we remark that, while not explicitly proved in [37], the result in Theorem 4.7 extends to higher dimensions with no change other that the role of the energy is played by the critical Sobolev norm of the initial data. However, the analogue of Theorem 3.7 is not true as stated. Instead we have the following

Open Problem 4.17. *Consider wave maps in dimension $n \geq 3$ with uniformly bounded critical Sobolev norms.*

a) Identify all possible concentration scenarios (at the very least, this must include solitons and self-similar solutions)

b) Establish a dichotomy as in Theorem 3.7 between energy dispersion and concentration scenarios.

Next we return to the two dimensional case. In the results above we have considered solutions below the ground state energy. But what happens if we take data with size slightly above the ground state energy ? For simplicity we will discuss the special case of maps from $\mathbb{R}^{2+1}$ into $(M, g) = \mathbb{S}^2$. There we have the harmonic maps Q_k which are the unique energy minimizers in their homotopy class modulo symmetries[6]. Recall that the ground state $Q = Q_1$ is the stereographic projection.

Consider the wave map equation with data which is close in the energy norm to Q_k. Such data must be in the same homotopy class as Q_k, and the solution stays there as long as no blow-up occurs. Then, due to energy conservation, we conclude that the ground states are orbitally stable, i.e. the solution must stay close to Q_k modulo symmetries. However, this does not lead to a global result since the group of symmetries is noncompact. Precisely, it is the scaling that generates the noncompactness[7] and may lead to blow up.

A natural simplification is to look at equivariant solutions. Then all other components of the symmetry group are eliminated, and we are left only with scaling. Thus we are looking at solutions of the form

$$\phi = Q_k(\lambda r) + o(1) \tag{4.99}$$

where λ is some function of t. Blow-up at time t_0 would correspond to $\lambda(t) \to \infty$ as $t \to t_0$. Blow-up solutions have been proved to exist:

Theorem 4.18 (Krieger-Schlag-Tataru [21])**.** *Let $k = 1$. Then there exist equivariant blow-up solutions with the concentration rate*

$$\lambda(t) = t^{-\nu-1}, \quad \nu > 1 \tag{4.100}$$

---

[6]Namely, isometries of $\mathbb{R}^2$, rotations of $\mathbb{S}^2$ and scaling.

[7]The spatial translations are another source of noncompactness, but cannot lead to blow up because of the finite speed of propagation.

Theorem 4.19 (Rodnianski-Sterbenz [34], Raphael-Rodnianski [32]). *Let $k \geq 1$. Then there exist equivariant blow-up solutions with the concentration rate*

$$\lambda(t) = t^{-1}|\log t|^{-\frac{1}{2k-2}}, \qquad k \geq 2 \tag{4.101}$$

$$\lambda(t) = e^{c\sqrt{|\log t|}} \qquad k = 1 \tag{4.102}$$

We expect the first result to be true for all $\nu > 0$. The second result seems to be in some sense an extreme case. The proof of these results is strongly related to the linearized wave map flow around the ground states Q_k. There is a fundamental difference between the case $k = 1$ and $k \geq 2$. In the latter case, the linearized elliptic operator has a zero eigenvalue, which is the source of instability. In the former case, we have instead a zero resonance, which still leads to instability but in a more subtle way. A natural follow-up problem would be

Open Problem 4.20. *Classify all possible blow-up rates in the equivariant case, and study their stability.*

Is blow-up a generic phenomena or an atypical one ? The knowledge that we have so far seems to indicate that the following may be plausible at least for $k = 1$:

Conjecture 4.21. *Consider the equivariant wave map equation with data near the soliton Q_1. Then there exists a codimension one stable manifold of data separating the data set into two components, as follows:*
a) Data in one component leads to a shrinking soliton and to finite time blow up.
b) Data in the other component leads to an expanding soliton.

This picture may require small adjustments as more data becomes available. As in initial step, in recent work [5] we are able to construct a codimension two stable manifold.

It would also be very interesting to consider nonequivariant data:

Open Problem 4.22. *Classify all possible blow-ups for wave maps $\phi : \mathbb{R}^{2+1} \to \mathbb{S}^2$ with data near the ground state Q, in terms of a description akin to* (4.99)*, but with scaling replaced by all the symmetries, and with good asymptotics for the symmetry group parameters as functions of t near the blow-up time.*

5. Schrödinger maps

Here we consider Schrödinger maps $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{S}^2$, $n \geq 2$ and prove the small data result in Theorem 3.8. We recall that in n space dimensions the initial data belongs to the space $\dot{H}^{\frac{n}{2}}$. To keep the notations simple we will confine the discussion to the energy critical case $n = 2$; this is also the most difficult case. Beside the finite energy condition, it is technically convenient to assume that for some $Q \in \mathbb{S}^2$ we have

$$M(\phi) = \int_{\mathbb{R}^2} |\phi - Q|^2 \, dx < \infty$$

This is a conserved quantity. The size of M plays no role in any of the estimates. Its only purpose is to insure convergence to the constant state Q along the harmonic heat flow; this in turn is used in the construction of the caloric gauge. The use of this condition can be bypassed, but that is not pursued here.

5.1. **Frames and gauges.** The formulation we adopt for this problem uses the frame method. At each point (t,x) we consider an orthonormal frame (v,w) in $T_{\phi(t,x)}\mathbb{S}^2$, and use the complex representation of tangent vectors $X \to \langle X, v\rangle + i\langle X, w\rangle$. In particular we can express $\partial_m \phi$ in the (v,w) frame as

$$\psi_m = v \cdot \partial_m \phi + i\ w \cdot \partial_m \phi. \tag{5.1}$$

Here $m = \overline{1, n+1}$ where ψ_{n+1} corresponds to the time variable.

Given the frame coefficients

$$A_m = w \cdot \partial_m v. \tag{5.2}$$

we define the covariant differentiation operators

$$\mathbf{D}_m = \partial_m + iA_m$$

The differentiated variables ψ_k are subject to the compatibility conditions,

$$\mathbf{D}_m \psi_k = \mathbf{D}_k \psi_m \tag{5.3}$$

while the connection A_k satisfies the curvature conditions

$$\partial_l A_m - \partial_m A_l = \Im(\psi_l \overline{\psi_m}) = q_{lm}. \tag{5.4}$$

A direct computation shows that the Schrödinger map equation expressed in terms of the differentiated fields takes the form

$$\psi_{n+1} = i \sum_{l=1}^{n} \mathbf{D}_l \psi_l. \tag{5.5}$$

Using (5.3) and (5.4), it follows that for $m = 1, \dots, n$ we have

$$\mathbf{D}_t \psi_m = i \sum_{l=1}^{n} \mathbf{D}_l \mathbf{D}_l \psi_m + \sum_{l=1}^{d} q_{lm} \psi_l, \tag{5.6}$$

which is equivalent to

$$(i\partial_t + \Delta_x)\psi_m = -2i \sum_{l=1}^{n} A_l \partial_l \psi_m + \big(A_{d+1} + \sum_{l=1}^{n} (A_l^2 - i\partial_l A_l)\big)\psi_m - i \sum_{l=1}^{n} \psi_l \Im(\overline{\psi}_l \psi_m). \tag{5.7}$$

To view this as a self-contained system we need to make a gauge choice, which would uniquely determine the A_j's in terms of the ψ_j's. Ideally, we would like to have a gauge which would make the right hand side of the above equation perturbative. The analogy we have in mind here is with the cubic NLS problem. Indeed, in view of the relations (5.4) it is reasonable to assume that the A_j's are quadratic and higher order in ψ, therefore the right hand side above will only contain terms which are cubic and higher order.

The main difficulty primarily originates with the term

$$A_l \partial_l \psi_m,$$

which has an unfavorable balance of derivatives. Consider for instance the simplest gauge, namely the Coulomb gauge, which yields an expression of the form

$$D^{-1}(\psi\bar{\psi}) D\psi$$

This causes some difficulties in the case of $high \times high \to low$ interactions in the first factor; these can be resolved in high dimension ($n \geq 4$, see [3]), but the singularity at frequency zero is too strong in two and three dimensions.

This is what causes us to look for a different choice of gauge which avoids the above difficulty. A reason to hope that such a gauge might exist is given by the exact form of the right hand side in the equations (5.4). Precisely, the functions ψ_l and ψ_m there are not independent, instead they are connected via (5.3). This indicates that to the leading order, the expression $\psi_l \bar{\psi}_m$ is real when the two factors have equal frequencies. Such a cancellation is not at all captured by the Coulomb gauge. As it turns out, there is indeed a more favorable gauge, namely the caloric gauge. This was proposed in [42] in the context of the wave map equation, and then as a possible gauge for Schrödinger maps.

Precisely, at each time t we solve the harmonic heat flow equation with $\phi(t)$ as the initial data,

$$\left\{ \begin{array}{ll} \partial_s \tilde{\phi} = \Delta_x \tilde{\phi} + \tilde{\phi} \cdot \sum_{m=1}^{n} |\partial_m \tilde{\phi}|^2 & \text{on } [0,\infty) \times \mathbb{R}^d; \\ \tilde{\phi}(0,t,x) = \phi(t,x). & \end{array} \right. \tag{5.8}$$

We note that the Schrödinger map and the harmonic heat flow do not commute. Thus, the Schrödinger map equation is only valid at $s = 0$, and not for larger s.

We heuristically remark that as the heat time s approaches infinity, the solution $\phi(s)$ approaches the equilibrium state Q. This is related to our assumption that the "mass" of ϕ_0 is finite, and would not necessarily be true otherwise. This allows us to arbitrarily pick (v_∞, w_∞) at $s = \infty$ as an orthonormal base in $T_Q \mathbb{S}^2$, independently of t and x. To define the orthonormal frame (v, w) for all $s \geq 0$ we pull back (v_∞, w_∞) along the backward heat flow using parallel transport. This translates into the relation

$$w \cdot \partial_s v = 0 \tag{5.9}$$

Setting $\partial_0 = \partial_s$, we define the functions ψ_m and A_m for all $s \in [0,\infty)$ and $m = 0, \cdots, d+1$ by

$$\left\{ \begin{array}{l} \psi_m = v \cdot \partial_m \tilde{\phi} + i\ w \cdot \partial_m \tilde{\phi}; \\ A_m = w \cdot \partial_m v. \end{array} \right. \tag{5.10}$$

In addition, the parallel transport relation $w \cdot \partial_s v = 0$ yields the main gauge condition

$$A_0 = 0. \tag{5.11}$$

As in the case of the Schrödinger equation, a direct computation using the heat equation (5.8) and (5.3), (5.4) shows that

$$\psi_0 = \sum_{l=1}^{d} \mathbf{D}_l \psi_l. \tag{5.12}$$

Thus, using again (5.4), for any $m = 1, \dots, d+1$

$$\partial_0 \psi_m = \mathbf{D}_m \psi_0 = \sum_{l=1}^{d} \mathbf{D}_m \mathbf{D}_l \psi_l = \sum_{l=1}^{d} \mathbf{D}_l \mathbf{D}_m \psi_l + i \sum_{l=1}^{d} q_{ml} \psi_l$$
$$= \sum_{l=1}^{d} \mathbf{D}_l \mathbf{D}_l \psi_m + i \sum_{l=1}^{d} \Im(\psi_m \overline{\psi_l}) \psi_l,$$

which is equivalent to

heatcov2
$$(\partial_s - \Delta_x)\psi_m = 2i \sum_{l=1}^{d} A_l \partial_l \psi_m - \sum_{l=1}^{d} (A_l^2 - i\partial_l A_l)\psi_m + i \sum_{l=1}^{d} \Im(\psi_m \overline{\psi_l}) \psi_l. \tag{5.13}$$

On the other hand from (5.4) we obtain

$$\partial_s A_m = \Im(\psi_0 \overline{\psi_m}).$$

Then we can integrate back from $s = \infty$ to obtain

Aform
$$A_m(s) = -\int_s^\infty \Im(\psi_0 \overline{\psi_m})(r)\, dr = -\sum_{l=1}^{n} \int_s^\infty \Im\big(\overline{\psi_m}(\partial_l \psi_l + i A_l \psi_l)\big)(r)\, dr, \tag{5.14}$$

for any $m = 1, \dots, d+1$ and $s \in [0, \infty)$. Thus $A_{m|s=0}$ represents our choice of the gauge for the Schrödinger map equation. The reason we prefer the caloric gauge to the Coulomb gauge is the way the high-high frequency interactions are handled. Indeed, while in the Coulomb gauge the connection coefficients can be conceptually written in the form

$$A \approx \sum_{j<k} 2^{-k} P_j \psi P_k \psi + \sum_{j \le k} 2^{-j} P_j (P_k \psi P_k \psi),$$

substituting the first approximation $\psi(s) \approx e^{s\Delta}\psi(0)$ in (5.14) yields the relation

schem
$$A \approx \sum_{j<k} 2^{-k} P_j \psi P_k \psi + \sum_{j \le k} 2^{-k} P_j (P_k \psi P_k \psi). \tag{5.15}$$

This has a better frequency factor in the high $\times$ high $\to$ low frequency interactions.

5.2. **Function spaces.** To motivate our choice of spaces, recall the Schrödinger nonlinearities, see (5.7)

Schnonlin
$$L_m = -2i \sum_{l=1}^{d} A_l \partial_l \psi_m + \big(A_{d+1} + \sum_{l=1}^{d} (A_l^2 - i\partial_l A_l)\big)\psi_m - i \sum_{l=1}^{d} \psi_l \Im(\overline{\psi_l} \psi_m). \tag{5.16}$$

We would like to analyze these nonlinearities perturbatively in suitable spaces. The main difficulty is caused by the magnetic terms $-2i \sum_{l=1}^{d} A_l \partial_l \psi_m$. Using (5.15) (for simplicity consider only the terms corresponding to $k = j$) they can be written schematically in the form

schem2
$$\sum_{k,k' \in \mathbb{Z}} 2^{-k} P_k \psi P_k \psi \cdot 2^{k'} P_{k'} \psi. \tag{5.17}$$

If $k > k'$ then this is a Strichartz type term, but if $k < k'$ then we need to recover a full derivative at frequency k'. The way to do that is by using the lateral energy spaces $L_e^{\infty,2}$

associated to Schrödinger waves with a suitable angular localization in a lateral frame with direction e. These and more generally the $L_e^{p,q}$ spaces are defined as

$$L_e^{p,q} = L^p_{x_e} L^q_{t,x'_e}$$

where $(x_e = x \cdot e, x'_e)$ is the orthogonal frame associated to the direction e.

Then the above expression (5.17) [schem2] needs to be estimated in a dual space $L_e^{1,2}$. For this to work it would appear that we need to bound $P_k\psi$ in $L_e^{2,\infty}$. This estimate is valid in dimensions three and higher. However, in two space dimensions this is precisely the forbidden endpoint of the (lateral) Strichartz estimates.

Nevertheless, the corresponding L^2 bilinear estimate for free Schrödinger waves is valid,

$$\|\psi_k \psi_{k'}\|_{L^2} \lesssim 2^{\frac{k-k'}{2}} \|\psi_k(0)\|_{L^2} \|\psi_{k'}(0)\|_{L^2}, \qquad k < k'$$

This suggests that there might be a way to still close by more subtle adjustments to the function spaces. The key observation which allows us to fix the above argument in two space dimensions is that in the lateral energy spaces $L_e^{\infty,2}$ used at frequency k' we are free to add Galilean transformations T_v as long as $|v| \ll 2^{k'}$. Here

$$T_v \phi(x,t) = e^{-i(\frac{1}{2}xv + \frac{1}{4}|v|^2 t)} \phi(x+vt,t)$$

In other words we can set

$$\|\phi\|_{L^{p,q}_{e,v}} = \|T_v\phi\|_{L^{p,q}_e}$$

and work with the smaller space

$$\bigcap_{|v| \ll 2^{k'}} L^{\infty,2}_{e,v}$$

This would allow us to relax the bound for $P_k\psi$ to the space

$$\sum_{|v|\approx 2^k} L^{2,\infty}_{e,v}$$

This strategy actually works. Furthermore, we do not need to use all such v, it suffices to restrict our attention to those which are parallel to e. In addition, by restricting time to a large but finite interval, we can discretize the above continuous set of v's. Precisely, for large $\mathcal{K}$ we restrict time to $t \in [0, 2^{2\mathcal{K}}]$, and then define the set of indices

$$W_k = W_k(\mathcal{K}) = \{\lambda \in [-2^k, 2^k] : 2^{k+2\mathcal{K}}\lambda \in \mathbb{Z}\}.$$

and the associated space

$$L^{2,\infty}_{e,W_k} = \sum_{v\in eW_k} L^{2,\infty}_{e,v}$$

To use these spaces we need the projectors $P_{k,e}$ which select the region $|\xi \cdot e| \approx 2^k$. Then we have the following

Lemma 5.1. *Let $d = 2$. For any $f \in L^2$, $k \in \mathbb{Z}$, and $e \in \mathbb{S}^1$ we have*

litl (5.18)
$$\|e^{it\Delta} P_{k,e} f\|_{L^{\infty,2}_{e,v}} \lesssim 2^{-k/2} \|f\|_{L^2}, \qquad |v| \ll 2^k.$$

In addition, if $T \in (0, 2^{2\mathcal{K}}]$ then

ltil (5.19)
$$\|1_{[-T,T]}(t) e^{it\Delta} P_k f\|_{L^{2,\infty}_{e,W_{k+40}}} \lesssim 2^{k/2} \|f\|_{L^2}.$$

l4s

Proof. We begin with (5.18). After a Galilean transformation the problem reduces to the case $v = 0$, where by translation invariance it suffices to estimate

$$\|u\|_{L^2_{t,x'}} \lesssim 2^{-\frac{k}{2}} \|f\|_{L^2}, \qquad u = e^{it\Delta} P_{k,e} f(t, 0, x'_e)$$

Without any restriction in generality we assume that $P_{k,e}$ is confined to the positive side $\xi \cdot e \approx 2^k$ (and not -2^k). Then a direct computation shows that

$$\hat{u}(\tau, \xi'_e) = \frac{1}{2\xi_e} p_{k,e}(\xi) \hat{f}(\xi), \qquad \tau = \xi^2, \quad \xi_e > 0$$

Hence (5.18) follows by a simple change of coordinates in the integral defining the L^2 norm.

Next we prove (5.19). For that we define two more classes of spaces. Given a finite subset $W \subseteq \mathbb{R}$ and $r \in [1, \infty]$ we define the spaces $\sum^r L^{p,q}_{e,W}$ and $\bigcap^r L^{p,q}_{e,W}$ using the norms

sumspacesr
$$\|\phi\|^r_{\sum^r L^{p,q}_{e,W}} = |W|^{r-1} \inf_{\phi = \sum_{\lambda \in W} \phi_\lambda} \sum_{\lambda \in W} \|\phi_\lambda\|^r_{L^{p,q}_{e,\lambda}} \tag{5.20}$$

and

intspacesr
$$\|\phi\|^r_{\bigcap^r L^{p,q}_{e,W}} = |W|^{-1} \sum_{\lambda \in W} \|\phi\|^r_{L^{p,q}_{e,\lambda}}. \tag{5.21}$$

Clearly, $\sum^1 L^{p,q}_{e,W} = L^{p,q}_{e,W}$ and

compr
$$\|\phi\|_{\sum^r L^{p,q}_{e,W}} \leq \|\phi\|_{\sum^{r'} L^{p,q}_{e,W}} \qquad \text{if } r \leq r'. \tag{5.22}$$

We fix $e \in \mathbb{S}^1$. By rescaling we can assume that $\mathcal{K} = 0$. We may also assume that $k \geq 1$, since for $k \leq 0$ one has the stronger bound

$$\|\mathbf{1}_{[-1,1]}(t) e^{it\Delta} P_k f\|_{L^2_x L^\infty_t} \lesssim \|f\|_{L^2}.$$

We need to show that

keymax
$$\|1_{[-1,1]}(t) e^{it\Delta} P_k f\|_{\sum^2 L^{2,\infty}_{e,W_{k+5}}} \lesssim 2^{k/2} \|f\|_{L^2}. \tag{5.23}$$

Due to the duality relation[8]

$$\Big(\bigcap^2 L^{2,1}_{e,W_{k+5}}\Big)' = \sum^2 L^{2,\infty}_{e,W_{k+5}}.$$

it suffices to show that if $\|g\|_{\bigcap^2 L^{2,1}_{e,W_{k+5}}} \leq 1$ then

keymax2
$$\Big| \int_{\mathbb{R}^2 \times \mathbb{R}} \overline{g(x,t)} \mathbf{1}_{[-1,1]}(t) (e^{it\Delta} P_k f)(x,t)\, dx dt \Big| \lesssim 2^{k/2} \|f\|_{L^2}. \tag{5.24}$$

This can be rewritten as

$$\Big| \int_{\mathbb{R}^2 \times \mathbb{R}} \overline{(e^{-it\Delta} P_k g(t))(x)} \mathbf{1}_{[-1,1]}(t) f(x)\, dt dx \Big| \lesssim 2^{k/2} \|f\|_{L^2}.$$

or equivalently

$$\Big\| \int_{-1}^{1} e^{-it\Delta} P_k g(t) \Big\|_{L^2} \lesssim 2^{k/2}$$

[8]This is not entirely straightforward.

Hence by a TT^* argument it suffices to show that

keymax3
$$\Big| \int_{\mathbb{R}^2\times\mathbb{R}} \int_{\mathbb{R}^2\times\mathbb{R}} g(x,t)\mathbf{1}_{[-1,1]}(t)\overline{g}(y,s)\mathbf{1}_{[-1,1]}(s) K_k(x-y,t-s)\, dxdtdyds \Big| \lesssim 2^k \tag{5.25}$$

where

K_kdef
$$K_k(x,t) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{-it|\xi|^2} \chi_k(|\xi|)^2 \, d\xi. \tag{5.26}$$

By stationary phase

$$|K_k(t,x)| \lesssim \begin{cases} 2^{2k}(1+2^{2k}|t|)^{-1} & |x| \le 2^{k+4}|t|; \\ 2^{2k}(1+2^k|x|)^{-N} & |x| \ge 2^{k+4}|t|. \end{cases}$$

The key idea is to foliate K_k in the e direction with respect to (thickened) rays with speed less than 2^{k+5}. We observe that for $t \in [-2,2]$

$$|K_k(t,x)| \lesssim \sum_{\lambda\in W_{k+5}} K_{k,\lambda}(t,x), \qquad K_{k,\lambda}(t,x) = (1+2^k|x\cdot e - \lambda t|)^{-N}.$$

Hence the left hand side of (keymax3 5.25) can be bounded by

$$\sum_{\lambda\in W_{k+5}} \int_{-1}^{1} \int_{-1}^{1} K_{k,\lambda}(t-s,x-y)|g(y,s)||g(x,t)| dxdydsdt$$
$$\lesssim \sum_{\lambda\in W_{k+5}} \|K_{k,\lambda}\|_{L^{1,\infty}_{e,\lambda}} \|g\|_{L^{2,1}_{e,\lambda}} \|g\|_{L^{2,1}_{e,\lambda}} \lesssim 2^{-k} \sum_{\lambda\in W_{k+5}} \|g\|^2_{L^{2,1}_{e,\lambda}} \lesssim 2^k \|g\|^2_{\bigcap^2 L^{2,1}_{e,W_{k+5}}},$$

where we used the fact that $|W_{k+5}| \approx 2^{2k}$. Thus (keymax2 5.24) follows. □

We are now ready to define the dyadic function spaces where we want to study the equation (schcov2 5.7). We will denote by G_k the spaces for the solutions ψ_m and by N_k the spaces for the right hand side L_m. Heuristically the G_k norms should contain Strichartz type norms, plus the above $\bigcap_{|v|\ll 2^k} L^{\infty,2}_{e,v}$ and the sum space $L^{\infty,2}_{e,W_k}$.

One difficulty we encounter is that the norms of nearby G_k's are not equivalent, and that makes it difficult to propagate them along the harmonic heat flow. For this reason we introduce a third space F_k with a weaker topology than G_k, $G_k \subset F_k$, but which does vary nicely with respect to k.

For comparison purposes, we also provide the corresponding definitions in dimensions three and higher.

spacesd>2 **Definition 5.2.** *Assume $n \ge 3$ and $k \in \mathbb{Z}$. Then $F_k(T)$, $G_k(T)$ and $N_k(T)$ are the Banach spaces of functions localized at frequency 2^k for which the corresponding norms are finite:*

fkdef3
$$\|\phi\|_{F_k(T)} = \|\phi\|_{L^\infty_t L^2_x} + \|\phi\|_{L^{p_d}} + 2^{-kd/(d+2)}\|\phi\|_{L^{p_d}_x L^\infty_t} + 2^{-k(d-1)/2} \sup_{e\in\mathbb{S}^{d-1}} \|\phi\|_{L^{2,\infty}_e}, \tag{5.27}$$

gkdef3
$$\|\phi\|_{G_k(T)} = \|\phi\|_{F_k} + 2^{k/2} \sup_{|j-k|\le 20} \sup_{e\in\mathbb{S}^{d-1}} \|P_{j,e}\phi\|_{L^{\infty,2}_e}, \tag{5.28}$$

respectively

nkdef3
$$\|f\|_{N_k(T)} = \inf_{f=f_1+f_2} \left(\|f_1\|_{L^{p_d'}} + 2^{-k/2} \sup_{e\in\mathbb{S}^{d-1}} \|f_2\|_{L^{1,2}_e} \right). \tag{5.29}$$

spacesd2 **Definition 5.3.** *Assume that $n=2$, $k\in\mathbb{Z}$, $\mathcal{K}\in\mathbb{Z}_+$, and $T\in(0,2^{2\mathcal{K}}]$. For functions ϕ at frequency 2^k let*

fkodef2 (5.30)
$$\|\phi\|_{F_k^0(T)}=\|\phi\|_{L_t^\infty L_x^2}+\|\phi\|_{L^4}+2^{-k/2}\|\phi\|_{L_x^4L_t^\infty}+2^{-k/2}\sup_{e\in\mathbb{S}^1}\|\phi\|_{L^{2,\infty}_{e,W_{k+40}}}.$$

We define $F_k(T)$, $G_k(T)$ and $N_k(T)$ as the spaces of functions for which the corresponding norms are finite:

fkdef2 (5.31)
$$\|\phi\|_{F_k(T)}=\inf_{J,m_1,\ldots,m_J\in\mathbb{Z}_+}\inf_{f=f_{m_1}+\ldots+f_{m_J}}\sum_{j=1}^J 2^{m_j}\|f_{m_j}\|_{F^0_{k+m_j}},$$

gkdef2 (5.32)
$$\begin{aligned}\|\phi\|_{G_k(T)}=\|\phi\|_{F_k^0}&+2^{-k/6}\sup_{e\in\mathbb{S}^1}\|\phi\|_{L_e^{3,6}}+2^{k/6}\sup_{|j-k|\le 20}\sup_{e\in\mathbb{S}^1}\|P_{j,e}\phi\|_{L_e^{6,3}}\\&+2^{k/2}\sup_{|j-k|\le 20}\sup_{e\in\mathbb{S}^1}\sup_{|\lambda|<2^{k-40}}\|P_{j,e}\phi\|_{L^{\infty,2}_{e,\lambda}},\end{aligned}$$

respectively

nkdef2 (5.33)
$$\|f\|_{N_k(T)}=\inf_{f=f_1+f_2+f_3+f_4}\left(\|f_1\|_{L^{\frac43}}+2^{\frac k6}\|f_2\|_{L_{e_1}^{\frac32,\frac65}}+2^{\frac k6}\|f_3\|_{L_{e_2}^{\frac32,\frac65}}+2^{-\frac k2}\sup_{e\in\mathbb{S}^1}\|f_4\|_{L^{1,2}_{e,W_{k-40}}}\right),$$

where (e_1,e_2) is the canonical basis in $\mathbb{R}^2$.

In all dimensions $d\ge 2$ the spaces $N_k(T)$ and $G_k(T)$ and related by the following linear estimate:

earmainrep **Proposition 5.4.** *(Main linear estimate) Assume $\mathcal{K}\in\mathbb{Z}_+$, $T\in(0,2^{2\mathcal{K}}]$ and $k\in\mathbb{Z}$. Then for each $u_0\in L^2$ which is localized at frequency 2^k and any $h\in N_k(T)$ the solution u to*

$$(i\partial_t+\Delta_x)u=h,\qquad u(0)=u_0$$

satisfies

$$\|u\|_{G_k(T)}\lesssim\|u(0)\|_{L_x^2}+\|h\|_{N_k(T)}$$

To bound products of functions in $F_k(T)$ we often use a more relaxed criterion. Precisely, since for $e\in\mathbb{S}^1$ and f localized at frequency 2^k we have

$$\|f\|_{L^{2,\infty}_{e,W_{k+m_j}}}\le\|f\|_{L_e^{2,\infty}}\lesssim 2^{k(d-1)/2}\|f\|_{L_x^2L_t^\infty}$$

it follows that, in all dimensions $d\ge 2$,

useboundin (5.34)
$$\|f\|_{F_k(T)}\lesssim\|f\|_{L_x^2L_t^\infty}+\|f\|_{L^{p_d}}.$$

This criterion is often used to estimate bilinear expressions, by exploiting the $L_x^{p_d}L_t^\infty$ norms in the spaces $F_k(T)$.

We also need to evolve $F_k(T)$ functions along the heat flow. Since the $F_k(T)$ norm is translation invariant it immediately follows that if $h\in F_k(T)$ then

hebound (5.35)
$$\|e^{s\Delta_x}h\|_{F_k(T)}\lesssim(1+s2^{2k})^{-20}\|h\|_{F_k(T)},\qquad s\ge 0.$$

To prove useful bounds on the connection coefficients A_m, $m=1,\ldots,d$, for $k\in\mathbb{Z}$ and $\omega\in[0,1/2]$ we define the normed spaces $S_k^\omega(T)$ of functions in $L_k^2(T)$ for which

sk (5.36)
$$\|f\|_{S_k^\omega(T)}=2^{k\omega}(\|f\|_{L_t^\infty L_x^{2\omega}}+\|f\|_{L_t^{p_d}L_x^{p_{d,\omega}}}+2^{-kd/(d+2)}\|f\|_{L_x^{p_{d,\omega}}L_t^\infty})<\infty,$$

where the exponents 2_ω and $p_{d,\omega}$ are such that

$$\frac{1}{2_\omega} - \frac{1}{2} = \frac{1}{p_{d,\omega}} - \frac{1}{p_d} = \frac{\omega}{d}.$$

The spaces $S_k^\omega(T)$ are at the same scale as the spaces $F_k(T)$ and $F_k(T) \hookrightarrow S_k^0(T)$. By Sobolev embeddings we have

skin

$$\|f\|_{S_k^{\omega'}(T)} \lesssim \|f\|_{S_k^\omega(T)} \quad \text{if } \omega' \leq \omega. \tag{5.37}$$

Thus the spaces $S_k^\omega(T)$ can be interpreted as refinements of the Strichartz part of the spaces $F_k(T)$ (which corresponds to $S_k^0(T)$). It is important to be able to prove bounds on the coefficients A_m, $m = 1, \ldots, d$, in both spaces $F_k(T)$ and $S_k^{1/2}(T)$. These bounds quantify an essential gain of smoothness of the coefficients A_m compared to the fields ψ_m.

5.3. **The small data result.** Here we outline the main steps in the proof of the small data result for Schrödinger maps in Theorem 3.8.

5.3.1. *Bounds for the harmonic heat flow.* We begin with the L^2 bounds for the harmonic heat flow. Below we state them for small data, but by the work of Smith [36] similar results hold up to the critical energy E_{crit}. For the next result we fix the Schrödinger time:

TaoHeat

Proposition 5.5. *(Construction of the caloric gauge) Let $\phi : \mathbb{R}^n \to \mathbb{S}^2$ with $\phi - Q \in L^2$ which satisfies the smallness condition*

smallnorm

$$\|\phi\|_{\dot H^{\frac{n}{2}}} = \gamma^2 \ll 1 \tag{5.38}$$

Let c_k be a frequency envelope for ϕ. Then there is a unique smooth solution $\tilde\phi \in C^\infty((0,\infty)\times \mathbb{R}^n)$ of the covariant heat equation

heat3a

$$\begin{cases} \partial_s \tilde\phi = \Delta_x \tilde\phi + \tilde\phi \cdot \sum_{m=1}^d |\partial_m \tilde\phi|^2 & \text{on } [0,\infty)\times\mathbb{R}^d; \\ \tilde\phi(0,x,t) = \phi(x,t). \end{cases} \tag{5.39}$$

In addition, there are smooth functions $v, w : [0,\infty)\times\mathbb{R}^d \to \mathbb{S}^2$ with the properties

vwprop

$$v\cdot\tilde\phi = w\cdot\tilde\phi = v\cdot w = w\cdot\partial_s v = 0 \ \text{ on } [0,\infty)\times\mathbb{R}^d\times(-T,T), \tag{5.40}$$

and for any $F \in \{\tilde\phi, v, w\}$ we have the bounds

oodbounds1

$$\|P_k F(s)\|_{L_x^2} \lesssim c_k (1 + s2^{2k})^{-20} 2^{-\frac{n}{2}k} \tag{5.41}$$

The key caloric gauge condition is the last identity in (5.40), namely ${}^t w \cdot \partial_s v \equiv 0$, which leads to the identity $A_0 \equiv 0$. It is also important that the functions $\tilde\phi, v, w$ become trivial as $s \to \infty$.

The L^2 bounds are far from sufficient for our analysis. Instead we need additional F_k bounds for the harmonic heat flow. This happens at the level of space-time estimates, so we add a Schrödinger time variable back into the picture. Again it is convenient to add the frequency envelopes to this picture. This is done with respect to the F_k norm. Thus, let c_k be an F_k frequency envelope for the ψ_m's. To this envelope we associate the sequence

$$c_{>k} = \Big(\sum_{j\geq k} c_j^2\Big)^{1/2}$$

TaoHeat2 **Proposition 5.6.** *(Heat flow bootstrap estimates) For $T \in (0,\infty)$ and ϕ small in $L^\infty \dot H^1(T)$ we consider $\tilde\phi, v, w$ as in Proposition 5.5, and ψ_m and A_m the associated fields and connection coefficients.*

(a) Suppose that the functions $\{\psi_m\}_{m=1,d}$ satisfy

psizero
$$\|P_k \psi_m(0)\|_{F_k(T)} \leq 2^{-k(d-2)/2} c_k, \qquad \epsilon := \|c\|_{l^2} \ll 1 \tag{5.42}$$

as well as the bootstrap condition

psibound
$$\|P_k \psi_m(s)\|_{F_k(T)} \leq \epsilon^{-1/2} c_k 2^{-k(d-2)/2} (1+s2^{2k})^{-4}. \tag{5.43}$$

Then we have

psifk
$$\|P_k \psi_m(s)\|_{F_k(T)} \lesssim c_k 2^{-k(d-2)/2} (1+s2^{2k})^{-4} \tag{5.44}$$

Also, for $l, m = 1, \cdots, n$ we have the $F_k(T)$ bounds

apsifk
$$\|P_k(A_m(s)\psi_l(s))\|_{F_k(T)} \lesssim c_k 2^{-k(d-4)/2} (2^{2k}s)^{-\frac{3}{8}} (1+s2^{2k})^{-2}, \tag{5.45}$$

as well as the L^{p_d} estimate at $s=0$

pkam
$$\|P_k A_m(0)\|_{L^{p_d}} \lesssim c_k 2^{-k(d-2)/2} \tag{5.46}$$

(b) Assume in addition that

psidpu
$$\|P_k \psi_{d+1}(0)\|_{L^{p_d}} \lesssim c_k 2^{-k(d-4)/2} 2^k \tag{5.47}$$

Then we have

$$\|P_k \psi_{d+1}(s)\|_{L^{p_d}} \lesssim c_k 2^{-k(d-4)/2} (1+2^{2k}s)^{-2}, \tag{5.48}$$

and the connection coefficient A_{d+1} satisfies the L^2 estimate at $s=0$

aadunu
$$\|P_k A_{d+1}(0)\|_{L^2} \lesssim c_k 2^{-k(d-2)/2}, \qquad n \geq 3 \tag{5.49}$$

respectively

aadunub
$$\|A_{d+1}(0)\|_{L^2} \lesssim \epsilon^2, \qquad \|P_k A_{d+1}(0)\|_{L^2} \lesssim c^2_{>k} \qquad d=2. \tag{5.50}$$

heatfk

The bootstrap assumption (5.43) can be then eliminated.

5.3.2. *Bounds for the Schrödinger map flow.* Since the connection coefficients A_m are defined via the harmonic heat flow, we cannot use a direct fixed point argument in order to solve the Schrödinger map equation. Instead, we use a bootstrap argument. Our main Schrödinger bootstrap result is the following.

Proposition 5.7. *(Schrödinger bootstrap estimates) Assume that $T \in (0, 2^{2\mathcal{K}}]$ and $Q \in \mathbb{S}^2$. Let $\{c_k\}_{k\in\mathbb{Z}}$ be an ϵ-frequency envelope with $\epsilon \ll 1$. Let ϕ be a smooth Schrödinger map in $[0,T]$ whose initial data satisfies*

schid
$$\|P_k \nabla \phi_0\|_{L^2_x} \leq c_k 2^{-k(d-2)/2} \tag{5.51}$$

Assume that ϕ satisfies the bootstrap condition

bootso
$$\|P_k \nabla \phi\|_{L^\infty_t L^2_x} \leq \epsilon^{-1/2} c_k 2^{-k(d-2)/2} \tag{5.52}$$

and let (ϕ, v, w) be the caloric extension of ϕ given by Proposition 5.5, with the corresponding fields ψ_m, A_m. Suppose also that at the initial parabolic time $s = 0$ the functions $\{\psi_m\}_{m=1,d}$ satisfy the additional bootstrap condition

bootsch (5.53)
$$\|P_k\psi_m(0)\|_{G_k(T)} \leq \epsilon^{-1/2}2^{-(d-2)k/2}c_k.$$

Then we have

bootschout (5.54)
$$\|P_k\psi_m(0)\|_{G_k(T)} \lesssim 2^{-(d-2)k/2}c_k.$$

schgk

The above proposition is proved by applying the linear result in Proposition 5.4 to the equation (5.7). The right hand side in (5.7) is estimated in the $N_k(T)$ spaces using the bounds in Proposition 5.6 for the differentiated fields ψ_m and the connection coefficients A_m.

We note that the bootstrap assumption (5.53) is eliminated via a continuity argument. The additional bootstrap condition (5.52) can also be improved to

bootso-i (5.55)
$$\|P_k\nabla\phi\|_{L^\infty_t L^2_x} \lesssim c_k$$

and then eliminated, by first transferring it to v and w using Proposition 5.5, and then by recovering $\nabla\phi$ via the relations (5.1).

5.3.3. *Rough solutions and continuous dependence.* To define rough solutions and study the dependence of solutions on the initial data we consider the linearized Schrödinger map equation. Expressed in the frame, this has the form

schlin (5.56)
$$(i\partial_t + \Delta_x)\psi_{lin} = -2i\sum_{l=1}^{d} A_l\partial_l\psi_{lin} + \big(A_{d+1} + \sum_{l=1}^{d}(A_l^2 - i\partial_l A_l)\big)\psi_{lin} - i\sum_{l=1}^{d}\psi_l\Im(\overline{\psi_l}\psi_{lin}).$$

This can be derived by direct computations as before. Heuristically, one can also think of a one parameter family of solutions $\phi(h)$ for the Schrödinger map equation so that $\phi(0) = \phi$ and ψ_{lin} is the expression in the frame of $\partial_h\phi_{|h=0}$, and extend the frame (v, w) as h varies. For this we will prove that it is well-posed in $\dot{H}^{(d-2)/2}$.

Proposition 5.8. *Let ϕ be a Schrödinger map as above. Then for each initial data $\psi_{lin}(0) \in H^\infty$ there exists an unique solution $\psi_{lin} \in C(\mathbb{R}, H^\infty)$ for (5.56), which satisfies the bounds*

psl (5.57)
$$\sum_k 2^{(d-2)k}\|P_k\psi_{lin}\|^2_{G_k(T)} \lesssim \|\psi_{lin}(0)\|^2_{\dot{H}^{\frac{d-2}{2}}}$$

The proof of this result is identical to the proof of Proposition 5.6. As a consequence of this we obtain the Lipschitz dependence of solutions in terms of the initial data in a weaker topology:

Proposition 5.9. *Consider two initial data ϕ_0^0 and ϕ_0^1 in H^∞_Q which satisfy the smallness condition $\|\phi_0^h\|_{\dot{H}^{\frac{d}{2}}} \ll 1$, $h = 0, 1$, and let ϕ^0, ϕ^1 be the corresponding global solutions for (5.56). Then*

l2diff (5.58)
$$\sum_k 2^{(d-2)k}\|P_k(\phi^0 - \phi^1)\|^2_{L^\infty\dot{H}^{\frac{d-2}{2}}} \lesssim \|\phi_0^0 - \phi_0^1\|^2_{\dot{H}^{\frac{d-2}{2}}}$$

pdiff

To prove this, one needs to show that any two initial data ϕ_0^0 and ϕ_0^1 which are small in $\dot H^1$ can be joined with a one parameter family $\{\phi_0^h\}_{h\in[0,1]} \in C^\infty([0,1];H^\infty)$ of initial data so that:

twmref (5.59)
$$\int_0^1 \|\partial_h \phi_0^h\|_{\dot H^{\frac{d-2}{2}}} \approx \|\phi_0^0-\phi_0^1\|_{\dot H^{\frac{d-2}{2}}}$$

This was proved in [47].

The above proposition allows us to conclude the proof of the strong continuous dependence on the initial data. Precisely, we show that the data to solution map S_Q admits an unique continuous extension
$$S_Q : \dot H^{\frac{d}{2}} \cap \dot H_Q^{\frac{d-2}{2}} \to C(\mathbb{R}; \dot H^{\frac{d}{2}} \cap \dot H_Q^{\frac{d-2}{2}})$$
It suffices to consider a sequence of smooth initial data $\phi_0^n \in H_Q^\infty$ which satisfy uniformly the smallness condition $\|\phi_0^n\|_{\dot H^{\frac{d}{2}}} \ll 1$ and so that $\phi_0^n \to \phi_0$ in $\dot H^{\frac{d}{2}} \cap \dot H_Q^{\frac{d-2}{2}}$, and show that the corresponding sequence of global solutions is Cauchy in the space in $C(\mathbb{R}; \dot H^{\frac{d}{2}} \cap \dot H_Q^{\frac{d-2}{2}})$. By Proposition 5.9 it follows the ϕ^n is Cauchy in $C(\mathbb{R}; \dot H_Q^{\frac{d-2}{2}})$,

conta (5.60)
$$\lim_{n,m\to\infty} \|\phi^n-\phi^m\|_{C(\mathbb{R};\dot H^{\frac{d-2}{2}})} = 0$$

Consider frequency envelopes $\{c_k^n\}$ associated to ϕ_0^n. Since ϕ_0^n is convergent in $\dot H^{\frac{d}{2}}$ we can choose the corresponding envelopes $\{c_k^n\}$ to converge in l^2. Then we have the uniform summability property

contb (5.61)
$$\lim_{k_0\to\infty} \sup_n \sum_{k>k_0} (c_k^n)^2 = 0$$

Now we estimate
$$\begin{aligned}\|\phi^n-\phi^m\|^2_{C(\mathbb{R};\dot H^{\frac{d}{2}})} \leq & \|P_{\leq k_0}(\phi^n-\phi^m)\|^2_{C(\mathbb{R};\dot H^{\frac{d}{2}})} + \|P_{>k_0}\phi^n\|^2_{C(\mathbb{R};\dot H^{\frac{d}{2}})} + \|P_{>k_0}\phi^m\|^2_{C(\mathbb{R};\dot H^{\frac{d}{2}})} \\ \lesssim & \, 2^{k_0}\|P_{\leq k_0}(\phi^n-\phi^m)\|^2_{C(\mathbb{R};\dot H^{\frac{d-2}{2}})} + \sum_{k>k_0}(c_k^n)^2+(c_k^m)^2\end{aligned}$$
Hence using (5.60) we have
$$\limsup_{n,m\to\infty} \|\phi^n-\phi^m\|^2_{C(\mathbb{R};\dot H^{\frac{d}{2}})} \lesssim \sup_n \sum_{k>k_0}(c_k^n)^2$$
Letting $k_0\to\infty$, by (5.61) we obtain
$$\limsup_{n,m\to\infty} \|\phi^n-\phi^m\|_{C(\mathbb{R};\dot H^{\frac{d}{2}})} = 0$$
and the argument is concluded.

The continuity of the solution operator S_Q in higher Sobolev spaces
$$S_Q : \dot H^{\sigma} \cap \dot H_Q^{\frac{d-2}{2}} \to C(\mathbb{R}; \dot H^{\sigma} \cap \dot H_Q^{\frac{d-2}{2}}), \qquad \frac{d}{2} < \sigma \leq \sigma_1$$
can be obtained in the same manner.

5.4. **Further developments.**

5.4.1. *Other targets.* The frame method works well in the case of the $\mathbb{S}^2$ or $\mathbb{H}^2$ targets, but arbitrary Kahler targets are a different story. There the frame method would not yield a self contained system for the differentiated fields ψ_m.

Open Problem 5.10. *Prove small data well-posedness for the Schrödinger map equation with values into an arbitrary (say compact) Kahler manifold.*

5.4.2. *Large data.* For the purpose of this section we assume that we are in two space dimensions, i.e. the energy critical case. The reason for this is that in this case the energy is a meaningful invariant object which can be used in the description of the global behavior of solutions.

We begin with the case of the $\mathbb{H}^2$ target, where there are no finite energy harmonic maps, and no other known obstructions to global well-posedness. This is the geometric version of a defocusing problem. Then we have

Conjecture 5.11 (Defocusing Conjecture)**.** *Consider the Schrödinger map problem in two space dimensions, with values in $\mathbb{H}^2$. Then global well-posedness and scattering holds for all finite energy data.*

In the case of the $\mathbb{S}^2$ target, the harmonic maps provide an obvious obstruction to a large data result. In addition, scattering can only occur for solutions in the zero homotopy class. The smallest nontrivial soliton, on the other hand, is the stereographic projection, Q_1 which belongs to the homotopy one class. In order to emulate such a soliton in the zero homotopy class, one needs to wrap the sphere and then unwrap it; this requires twice the soliton energy. Thus the natural conjecture is:

Conjecture 5.12 (Strong Threshold Conjecture)**.** *Consider the Schrödinger map problem in two space dimensions, with values in $\mathbb{S}^2$. Then global well-posedness and scattering holds for all zero homotopy data which satisfies $E(\phi) < 2E(Q_1)$.*

These conjectures parallel recently proved results for wave maps. Both conjectures are still open for Schrödinger maps. However, the equivariant case has recently been studied.

Theorem 5.13 (Bejenaru-Kenig-Ionescu-Tataru, 2012arXiv1212.2566B [2])**.** *Consider the Schrödinger map problem in two space dimensions, with values in $\mathbb{H}^2$. For this problem, global well-posedness and scattering holds in the* 1*-equivariant class for all finite energy data.*

Theorem 5.14 (Bejenaru-Kenig-Ionescu-Tataru 2011arXiv1112.6122B [1])**.** *Consider the Schrödinger map problem in two space dimensions, with values in $\mathbb{S}^2$. For this problem, global well-posedness and scattering holds in the* 1*-equivariant class for all zero homotopy data which satisfies $E(\phi) < E(Q_1)$.*

The proof uses the Kenig-Merle method, which involves

- proving that if the result does not hold then minimal energy blow-up solutions exist and
- eliminating the minimal energy blow-up solutions via mass and momentum Morawetz type estimates.

Key difficulties in the proof:

- Gauge formulation of the problem: via the Coulomb gauge one obtains two coupled NLS type equations, and the coupling needs to survive in the concentration compactness argument.

- Morawetz (momentum) estimates are harder, and only yield local energy decay in a restricted regime; in particular we cannot reach the conjectured $2E(Q_1)$ threshold for $\mathbb{S}^2$ targets.

5.4.3. *Near soliton behavior.* In this section we consider the behavior of solutions with energy above the ground state threshold. For clarity we discuss only the simplest such problem, which is still wide open. Thus, we consider the case of the $\mathbb{S}^2$ target and solutions in the homotopy one class, which have energy just above the soliton energy,

nearQ (5.62)
$$E(Q_1) \leq E(\phi) < E(Q_1) + \epsilon$$

We note that if $E(Q_1) = E(\phi)$ then ϕ must belong to the class $\mathcal{Q}_1$ of ground states obtained from Q_1 via symmetries. We also remark that energy considerations show that any such state ϕ must satisfy

$$\mathrm{dist}(\phi, \mathcal{Q}_1) \lesssim \epsilon.$$

Thus the family $\mathcal{Q}_1$ is orbitally stable. Unfortunately this does not say as much as one might want since the group of symmetries is noncompact. Thus we have the following

Open Problem 5.15. *For Schrödinger maps from $\mathbb{R}^{2+1}$ to $\mathbb{S}^2$ which have homotopy one and satisfy* (nearQ 5.62)*, understand the possible global dynamics for the flow.*

The key element in this is understanding the motion of solutions along the Q_1 family. Possible issues to consider are

- Can finite time blow-up occur ? If so, what are the possible rates ?
- For global solutions, what is the asymptotic behavior at infinity (if any) ?
- Can solutions drift away to spatial infinity in finite time ? In infinite time ?
- Are there any breather type solutions in this class ?

While in such generality the above problem seems out of reach for now, some partial results have been obtained for equivariant solutions. An advantage of working in the equivariant class is that the dimension of the symmetry group is reduced to two, namely scaling and horizontal rotations. The first is noncompact, but the second is compact. Thus we can parametrize the ground states as

$$\mathcal{Q}_1^{eq} = \{Q_{\alpha,\lambda};\ \lambda \in \mathbb{R}^+, \alpha \in \mathbb{S}^1\}$$

The equivariant solutions are represented as

$$\phi(t) = Q_{\alpha(t),\lambda(t)} + O_{\dot{H}^1}(\epsilon)$$

and the question is to understand the behavior of the functions $\alpha(t)$ and $\lambda(t)$.

In chronological order, the results we have so far are as follows:

Theorem 5.16 (Gustafson-Nakanishi-Tsai MR2725187 [17])**.** *$\mathcal{Q}_k$ ground states are stable in the k equivariant class for $k \geq 3$.*

We remark that this result is very different from the wave-map picture. Also, it seems somewhat unlikely that the result will survive outside the equivariant class.

Theorem 5.17 (Bejenaru-Tataru ($k = 1$, 2010arXiv1009.1608B [6]) ($k = 2$, in progress))**.** *a) $\mathcal{Q}_1$ ground states are unstable in the energy norm $\dot{H}^1$.*

b) $\mathcal{Q}_1$ ground states are stable in the one equivariant class with respect to a stronger topology X satisfying

$$H^1 \subset X \subset \dot{H}^1$$

A key role in this analysis is played by the linearized equation near Q_1 expressed in a suitable gauge. This is a linear Schrödinger equation governed by an explicit operator

$$H = -\Delta + V, \qquad V(r) = \frac{1}{r^2} - \frac{8}{(1+r^2)^2}.$$

A key difficulty is that H has a zero resonance

$$\phi_0 = r\partial_r Q_1 = \frac{2r}{1+r^2}$$

which corresponds to motion along the soliton family.

This is unlike what happens in higher equivariance classes $k \geq 3$ where the analogue of ϕ_0 is not only an eigenvalue but also belongs to H^{-1}. This allows one to define a corresponding orthogonal projection for functions in $\dot{H}^1$ and opens the door to a more standard perturbation theory.

The proof of the above result requires developing a complete spectral resolution for the operator H. In addition, the parameter $\lambda(t)$ is the main nonperturbative parameter in this analysis, so one in effect needs to work with a linear evolution of the form

$$(i\partial_t + H_{\lambda(t)})\psi = f$$

with a nontrivial dependence of λ on t.

Finally, the last and most recent results in this direction that we mention are

Theorem 5.18 (Merle-Raphael-Rodnianski [26], Perelman [28])**.** *Finite time blow-up equivariant solutions exist near $\mathcal{Q}_1$.*

The first result [26] adapts to the Schrödinger map setting the techniques in the similar work for wave maps in [34], [32]. The second [28] is the Schrödinger map counterpart of the wave map results in [21].

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