(20) 1. Determine the interval of convergence of the following series. Do they converge at endpoints?

\[ a) \sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{\sqrt{n} \cdot 4^n} \]

Solution: Using the ratio test we compute
\[
\lim_{n \to \infty} \frac{\sqrt{n+1} \cdot 4^{n+1}}{(x-1)^{2n}} = \lim_{n \to \infty} \frac{(x-1)^2 \cdot \sqrt{n}}{4 \cdot \sqrt{n+1}} = \frac{(x-1)^2}{4}
\]
The limit is less than 1 if \( |x-1| < 2 \). Hence the radius of convergence is \( R = 2 \). At the endpoints we have \( x-1 = \pm 2 \) and the series becomes
\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \]
which is a divergent \( p \)-series. Thus the interval of convergence is \((-1, 3)\).

\[ b) \sum_{n=2}^{\infty} \ln \left( \frac{n+1}{n-1} \right) x^n \]

Solution: Using the Taylor expansion for \( \ln(1+x) \) we write
\[ \ln \left( \frac{n+1}{n-1} \right) = \ln \left( 1 + \frac{2}{n-1} \right) = \frac{2}{n-1} - \frac{2}{(n-1)^2} + \cdots \]
Then for the ratio test we compute
\[
\lim_{n \to \infty} \frac{\ln \left( \frac{n+2}{n} \right) x^{n+1}}{\ln \left( \frac{n+1}{n-1} \right) x^n} = x \lim_{n \to \infty} \frac{2 - \frac{2}{n} + \cdots}{2 - \frac{2}{n-1} + \cdots} = x
\]
The limit is less than 1 if \( |x| < 1 \). Hence the radius of convergence is \( R = 1 \). At the endpoint \( x = -1 \) we obtain the alternating series
\[ \sum_{n=2}^{\infty} (-1)^n \ln \left( 1 + \frac{2}{n-1} \right) \]
Due to the expansion above we have \( \ln \left( \frac{n+1}{n-1} \right) \downarrow 0 \) as \( n \to \infty \) therefore the series converges by the alternating test.
At the endpoint \( x = 1 \) we obtain the series
\[ \sum_{n=2}^{\infty} \ln \left( 1 + \frac{2}{n-1} \right) \]
Due to the expansion above this is comparable to the harmonic series \( \sum_{n=2}^{\infty} \frac{2}{n-1} \) which diverges.
Thus the interval of convergence is \([-1, 1)\).
Find the Maclaurin series expansion of the following functions. Determine where the expansions are valid (i.e. for what values of $x$ they converge).

a) $f(x) = \frac{x}{x^2 + x - 2}$

Solution: Using partial fractions we write

$$f(x) = \frac{x}{x^2 + x - 2} = \frac{x}{(x + 2)(x - 1)} = \frac{2}{3(x + 2)} + \frac{1}{3(x - 1)} = \frac{1}{3} \left( \frac{1}{1 + \frac{x}{2}} - \frac{1}{1 - x} \right)$$

Then using the geometric series we write

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n, \quad \frac{1}{1 + \frac{x}{2}} = \sum_{n=0}^{\infty} (-1)^n 2^{-n} x^n$$

Summing up we obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{3} \left( (-1)^n 2^{-n} - 1 \right) x^n$$

The radius of convergence is 1 for the first term and 2 for the second, so after adding them up we obtain $R = 1$. At the endpoints $x = \pm 1$ the series diverges since the general term does not go to 0. Hence the interval of convergence is $(-1, 1)$.

b) $f(x) = \sqrt{1 + x^2}$

Solution: We use the binomial series

$$\sqrt{1 + x} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right) \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{2} \frac{(1 - 1) \cdots (\frac{1}{2} - n + 1)}{n!} x^n = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{2n} \frac{1 \cdot 3 \cdots (2n - 3)}{2^n n!} x^n$$

and replace $x$ by $x^2$ to obtain

$$\sqrt{1 + x^2} = 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdots (2n - 3)}{2^n n!} x^{2n}$$

The binomial series converges for $|x| < 1$ therefore our series also converges for $|x| < 1$. This can also be verified directly using the ratio test. At the endpoints $x = \pm 1$ we obtain the alternating series

$$\sum_{n=0}^{\infty} (-1)^{n-1} a_n, \quad a_n = \frac{1 \cdot 3 \cdots (2n - 3)}{2^n n!}$$

It is easily verified that the sequence $a_n$ is decreasing, but harder to show that it converges to 0. We have

$$a_n = \frac{1}{2^n} \frac{2n - 3}{2n - 2} \cdot \frac{1}{2n} \cdot \frac{1}{2n} \to 0$$

This implies that $a_n \to 0$. Then the interval of convergence is $[-1, 1]$. 
3. a) Find the third order Taylor polynomial of \(\tan x\) at \(\pi/4\).

Solution: For \(f(x) = \tan x\) we compute

\[
f'(x) = \sec^2 x, \quad f''(x) = 2 \sec^2 x \tan x, \quad f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x
\]

We evaluate them at \(\pi/4\) using \(\tan \pi/4 = 1, \sec \pi/4 = \sqrt{2}\). This gives

\[
f(\pi/4) = 1, \quad f'(\pi/4) = 2, \quad f''(\pi/4) = 4, \quad f'''(\pi/4) = 16
\]

Then the third order Taylor polynomial of \(\tan x\) at \(\pi/4\) is

\[
P_3(x) = 1 + 2(x - \pi/4) + 2(x - \pi/4)^2 + 8/3(x - \pi/4)^3
\]

b) Find the Maclaurin series for a function \(f\) which solves the differential equation

\[
f''(x) = xf(x), \quad f(0) = 1, \quad f'(0) = 0
\]

What is the radius of convergence?

Solution: If \(f(x) = \sum_{n=0}^{\infty} a_n x^n\) then \(f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}\) therefore

\[
f'(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 1 \cdot 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + \cdots + (n+2)(n+1)a_{n+2}x^n + \cdots
\]

On the other hand

\[
xf(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = a_0 x + a_1 x^2 + \cdots + a_{n-1} x^n + \cdots
\]

Identifying the coefficients in the two power series we obtain \(a_2 = 0\) and

\[
(n+2)(n+1)a_{n+2} = a_{n-1}, \quad n \geq 1
\]

From the initial data we also know that \(a_0 = 1, a_1 = 0\). Then we can iteratively compute the coefficients \(a_n\) (e.g. we use the above formula with \(n = 1\) to compute \(a_3\), etc.):

\[
1, 0, 0, \frac{1}{2}, \frac{3}{3}, 0, 0, \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}, 0, 0, \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, \cdots
\]

This gives the Maclaurin series

\[
f(x) = \sum_{n=0}^{\infty} \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n - 1)3n} x^{3n}
\]

To compute the radius of convergence we use the ratio test. We have

\[
\lim_{n \to \infty} \frac{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n - 1)3n(3n + 2)(3n + 3)x^{3n+3}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n - 1)3nx^{3n}} = \lim_{n \to \infty} \frac{x^3}{(3n + 2)(3n + 3)} = 0
\]

Hence the series converges for all \(x\).
4. Sketch the direction field of
\[ y' = y^3 - y \]
and determine the equilibrium solutions. Are they stable?

Solution:

a) We check the sign of \( y' \):

\[
\begin{array}{c|ccc}
 y & -1 & 0 & 1 \\
\hline
 y' & - & + & - & 0 & + \\
\end{array}
\]

The equilibrium solutions are \( y = \pm 1 \) and \( y = 0 \).

b) We sketch the direction field (see the picture in problem 1b, Section 9.2 but with the \( x \) axis reversed)

c) Sketch a few solutions which follow the direction field. The solution \( y = 0 \) is stable, but \( y = \pm 1 \) are not.
5. Solve the initial value problems

\( a) \quad \frac{dx}{dt} = 2t(1 + x^2), \quad x(0) = 0 \)

Solution: This is a separable equation. We compute

\[
\frac{dx}{1 + x^2} = 2tdt, \quad \int \frac{dx}{1 + x^2} = \int 2tdt + C
\]

which gives

\[ \tan^{-1} x = t^2 + C \]

Using the initial data we obtain \( C = 0 \), therefore the solution is

\[ x(t) = \tan t^2 \]

\( b) \quad \frac{dx}{dt} = x + \sin t, \quad x(0) = 1 \)

This is a linear equation, which we rewrite as

\[ x' - x = \sin t \]

The integrating factor is \( e^{-t} \). Multiplying by it in both sides gives

\[ e^{-t}x - e^{-t}x = e^{-t}\sin t \quad \iff \quad (e^{-t}x)' = e^{-t}\sin t \]

Hence integrating by parts we obtain

\[ e^{-t}x(t) = \int e^{-t} \sin tdt = -\frac{1}{2}e^{-t}(\sin t + \cos t) + C \]

so the general solution is

\[ x(t) = -\frac{1}{2}(\sin t + \cos t) + Ce^t \]

Using the initial data in this equation gives \( C = \frac{3}{2} \), therefore

\[ x(t) = -\frac{1}{2}(\sin t + \cos t) + \frac{3}{2}e^t \]