1. Evaluate the following (indefinite) integrals 

a) $\int e^{\sqrt{x}} \, dx$

Solution: Substitute $x = u^2$, $dx = 2udu$. The integral becomes

$$\int 2ue^u \, du$$

We integrate by parts to obtain

$$\int 2ue^u \, du = 2ue^u - \int 2e^u \, du = 2ue^u - 2e^u + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

b) $\int x \tan^2 x \, dx$

Solution: We rewrite the integral as

$$\int x \tan^2 x \, dx = \int x(\sec^2 x - 1) \, dx = \int x \sec^2 x \, dx - \frac{x^2}{2}$$

The first term is integrated by parts,

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x + \ln |\cos x| + C$$

The final result is

$$\int x \tan^2 x \, dx = x \tan x + \ln |\cos x| - \frac{x^2}{2} + C$$
(20) 2. Evaluate the following (definite) integrals:

a) \[ \int_{-\infty}^{\infty} \frac{4x^2}{x^4 + 4} \, dx \]

Solution: We use partial fractions. First we factor the denominator,

\[ x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - (2x)^2 = (x^2 + 2x + 2)(x^2 - 2x + 2) \]

Then we decompose into partial fractions,

\[ \frac{4x^2}{x^4 + 4} = \frac{Ax + B}{x^2 - 2x + 2} - \frac{Cx + D}{x^2 + 2x + 2} \]

This gives

\[ 4x^2 = (Ax + B)(x^2 + 2x + 2) + (Cx + D)(x^2 - 2x + 2) \]

Identifying the coefficients we obtain the equations

\[ A + C = 0, \quad 2A + B - 2C + D = 4, \quad 2A + 2B + 2C - 2D = 0, \quad 2B + 2D = 0 \]

which has solutions \( A = 1, B = 0, C = -1, D = 0 \). Hence the indefinite integral becomes

\[ \int \frac{x}{x^2 - 2x + 2} - \frac{x}{x^2 + 2x + 2} \, dx = \int \frac{x - 1}{(x - 1)^2 + 1} + \frac{1}{(x + 1)^2 + 1} - \frac{x + 1}{(x + 1)^2 + 1} + \frac{1}{(x + 1)^2 + 1} \, dx \]

\[ = \frac{1}{2} \ln((x - 1)^2 + 1) + \tan^{-1}(x - 1) - \frac{1}{2} \ln((x + 1)^2 + 1) + \tan^{-1}(x + 1) + C \]

To find the definite integral we evaluate this between \( -\infty \) and \( \infty \). Since

\[ \lim_{x \to \pm\infty} \frac{(x - 1)^2 + 1}{(x + 1)^2 + 1} = 1 \]

we are left only with the contributions from the last two terms,

\[ \int_{-\infty}^{\infty} \frac{4x^2}{x^4 + 4} \, dx = [\tan^{-1}(x - 1) + \tan^{-1}(x + 1)]_{-\infty}^{\infty} = 2\pi \]

b) \[ \int_{0}^{\pi/2} \frac{\cos x}{\sqrt{1 + \sin^2 x}} \, dx \]

Solution: First substitute \( u = \sin x, \, du = \cos x \, dx \). The integral becomes

\[ \int_{0}^{1} \frac{1}{\sqrt{1 + u^2}} \, du \]

Then substitute \( u = \tan \theta, \, du = \sec^2 \theta \, d\theta \), transforming the integral into

\[ \int_{0}^{\pi/4} \frac{\sec^2 \theta}{\sec \theta} \, d\theta = \int_{0}^{\pi/4} \sec \theta d\theta = \ln |\sec \theta + \tan \theta|_{0}^{\pi/4} = \ln(\sqrt{2} + 1) \]
(20) 3. a) Suppose that \( f(x) \) is a function defined on \([a, b]\). State the formula for the area of the surface of revolution obtained by rotating the graph of \( f \) around the \( y \) axis.

Solution:
\[
A = 2\pi \int_a^b x \sqrt{1 + f'(x)^2} \, dx
\]

b) Find that area in the case when \( f(x) = 3x^{1/3} \) and \( a = 0, b = 1 \).

Solution: We have
\[
A = 2\pi \int_0^1 x \sqrt{1 + x^{-4/3}} \, dx
\]
Substituting \( x = u^3 \), \( dx = 3u^2 \, du \) we transform this into
\[
2\pi \int_0^1 3u^5 \sqrt{1 + u^{-4}} \, du = 2\pi \int_0^1 3u^3 \sqrt{u^4 + 1} \, du
\]
Setting \( u^4 + 1 = v \), \( 4u^3 \, du = dv \) the integral becomes
\[
A = \pi \int_1^2 \frac{3}{2} \sqrt{v} \, dv = \pi \left. v^{3/2} \right|_1^2 = \pi (2\sqrt{2} - 1)
\]
4. Determine (providing an explanation) the convergence or divergence of the following series:

a) \( \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}} \)

Solution: Use the integral test to compare with the integral

\[
\int_{2}^{\infty} \frac{1}{x\sqrt{\ln x}} \, dx
\]

Substituting \( \ln x = u \), \( x^{-1} \, dx = du \) the indefinite integral turns into

\[
\int \frac{1}{\sqrt{u}} \, du = 2\sqrt{u} + C = 2\sqrt{\ln x} + C
\]

Then for the improper integral we get

\[
\int_{2}^{\infty} \frac{1}{x\sqrt{\ln x}} \, dx = \lim_{b \to \infty} 2\sqrt{\ln b} - 2\sqrt{\ln 2} = \infty
\]

Hence the improper integral diverges. Then the series is also divergent.

b) \( \sum_{n=1}^{\infty} \frac{1 + (-1)^n n}{n^2 + 2n} \)

Solution: We split the series in two,

\[
\frac{1 + (-1)^n n}{n^2 + 2n} = \frac{1}{n^2 + 2n} + \frac{(-1)^n n}{n^2 + 2n}
\]

We have \( \frac{1}{n^2 + 2n} \leq \frac{1}{n^2} \) therefore the series \( \sum \frac{1}{n^2 + 2n} \) converges by comparison with the \( p \)-series.

On the other hand the series \( \sum \frac{(-1)^n n}{n^2 + 2} \) converges due to the alternating test.

Summing up the two series we conclude that the original series converges.

c) \( \sum_{n=1}^{\infty} \frac{(n!)^2}{e^{n^2}} \)

Solution: Use ratio test:

\[
\lim_{n \to \infty} \frac{\frac{(n+1)!}{e^{(n+1)^2}}}{\frac{n!}{e^{n^2}}} = \lim_{n \to \infty} \frac{(n+1)^2}{e^{2n+1}}
\]

We compute this limit using L’Hopital’s rule,

\[
\lim_{x \to \infty} \frac{(x+1)^2}{e^{2x+1}} = \lim_{x \to \infty} \frac{x+1}{e^{2x+1}} = \lim_{x \to \infty} \frac{1}{2e^{2x+1}} = 0
\]

By the ratio test it follows that the series is convergent.
5. a) Estimate the error in approximating the following series by the sum of its first 10 terms:

\[ \sum_{n=1}^{\infty} \frac{1}{n^4 + n^2} \]

Solution: We first estimate \( \frac{1}{n^4 + n^2} \leq \frac{1}{n^4} \). Since the function \( x^{-4} \) is decreasing, the error is estimated in terms of the integral,

\[ |R_n| \leq \int_n^{\infty} \frac{1}{x^4} \, dx = -\frac{1}{3x^3} \bigg|_n^{\infty} = \frac{1}{3n^3} \]

Hence

\[ |R_{100}| \leq \frac{1}{3000000} \]

b) Estimate the partial sums of the series

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \]

Solution: The series is a \( p \)-series which diverges. Since the function \( x^{-4} \) is decreasing, we can compare the partial sums with the corresponding integral,

\[ S_n \approx \int_1^n \frac{1}{\sqrt{x}} \, dx = \frac{1}{2}(\sqrt{n} - 1) \]

c) Compute the sum of the series

\[ \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \]

Using partial fractions we write

\[ \frac{1}{n^2 - 1} = \frac{1}{2} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) \]

Then the series is a telescopic sum. Its partial sums are

\[ S_n = \frac{1}{2} \left( \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} \right) \]

Almost all terms cancel, and we obtain

\[ S_n = \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \right) \to \frac{3}{4} \]

Hence the sum of the series is \( 3/4 \).