A Hodge-theoretic study of augmentation varieties associated to Legendrian knots/tangles

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Abstract

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In this article, we give a tangle approach in the study of Legendrian knots in the standard contact three-space. On the one hand, we define and construct Legenrian isotopy invariants including ruling polynomials and Legendrian contact homology differential graded algebras (LCH DGAs) for Legendrian tangles, generalizing those of Legendrian knots. Ruling polynomials are the Legendrian analogues of Jones polynomials in topological knot theory, in the sense that they satisfy the composition axiom.

On the other hand, we study certain aspects of the Hodge theory of the "representation varieties (of rank 1)" of the LCH DGAs, called augmentation varieties, associated to Legendrian tangles. The augmentation variety (with fixed boundary conditions), hence its mixed Hodge structure on the compactly supported cohomology, is a Legendrian isotopy invariant up to a normalization. This gives a generalization of ruling polynomials in the following sense: the point-counting/weight (or E-) polynomial of the variety, up to a normalized factor, is the ruling polynomial. This tangle approach in particular provides a generalization and a more natural proof to the previous known results of M.Henry and D.Rutherford. It also leads naturally to a ruling decomposition of this variety, which then induces a spectral sequence converging to the MHS. As some applications, we show that the variety is of Hodge-Tate type, show a vanishing result on its cohomology, and provide an example-computation of the MHSs.

Dedicated to my parents.

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Chapter 0

Introduction

Similar to smooth knot theory, there's a parallel study of Legendrian knots in contact three manifolds. The fundamental case is the Legendrian knots in the standard contact three space. The classical Legendrian invariants are the topological knot type, the Thurston-Bennequin number and the rotation number. They determine a complete set of invariants for some Legendrian knots, including the unknots, torus knots and the figure eight knots [9, 10]. However, in general they do not determine a complete Legendrian knot invariant, as shown by the Chekanov pairs [4]. They have the same classical invariants, but are distinguished by a stronger invariant, the Chekanov-Eliashberg differential graded algebra. The Chekanov-Eliashberg DGAs are special cases of Legendrian contact homology differential graded algebras (LCH DGAs). Morally, associate to any pair (V, Λ) of a Legendrian submanifold Λ contained in a contact manifold V the LCH DGAs $(\mathcal{A}(V, \Lambda), \partial)$ are defined via Floer theory [7, 8]. The generators are indexed by the Reeb chords of Λ. The differential counts holomorphic disks in the symplectization $\mathbb{R}\times V$, with boundaries along the Lagrangian cylinder $\mathbb{R}\times\Lambda$, and meeting the Reeb chords at positive or negative infinity. The LCH DGAs are Legendrian isotopy invariants, up to homotopy equivalence. In the case of Legendrian knots Λ in the standard contact three space, the LCH DGA $(\mathcal{A}(\Lambda), \partial)$ can also be defined purely combinatorially [4, 11].

To extract some numerical invariants from the LCH DGAs, one fundamental idea in [4] is to consider the functor of points of $(\mathcal{A}(\Lambda), \partial)$:

$$\text{commutative ring } \mathbf{r} \to \{\text{DGA morphisms } (\mathcal{A}, \partial) \to \mathbf{r}\}/\text{DGA homotopy}$$

and count the points appropriately over a finite field. One way to do so is as follows. Let r be the gcd of the rotation numbers of the connected components of Λ, which ensures the existence of a $\mathbb{Z}/2r$-valued Maslov potential μ, the DGA $\mathcal{A}(\Lambda)$ associated to (Λ, μ) is then naturally $\mathbb{Z}/2r$-graded. Start with a nonnegative integer m dividing $2r$, we consider the space of $\mathbb{Z}/m$-graded augmentations ("m-graded points") valued in any finite field k. This defines an algebraic variety $\mathrm{A}ug_m(\Lambda, k)$, called the augmentation variety. Now, a normalized count of the points of $\mathrm{A}ug_m(\Lambda, \mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ gives the augmentation number

$aug_m(\Lambda, q) := q^{-\dim_{\mathbb{C}} Aug(\Lambda,\mathbb{C})}|Aug(\Lambda, \mathbb{F}_q)|$. These define Legendrian isotopy invariants [15, Thm.3.2] and in fact distinguish the Chekanov pairs.

More recently, some categorical Legendrian isotopy invariants, the augmentation categories $\mathcal{A}ug_+(\Lambda, k), \mathcal{A}ug_-(\Lambda, k)$, and also some of their equivalent versions or generalizations are constructed [31, 2, 26]. The augmentation categories are A_∞ categories, and up to A_∞-equivalence, are invariants under Legendrian isotopy of Λ. They can be viewed as the categorical refinement of augmentation varieties, in the sense that augmentation varieties only encodes the 0-th order information (points) of the LCH DGA $\mathcal{A}(\Lambda)$, while the augmentation categories also encode the higher order information (tangent spaces with additional structures). It's expected that, a refined counting of points using the augmentation category (homotopy cardinality) may give a more natural way to count augmentations (See [27]). It's likely that the tangle approach studied in this article may also provide a natural approach to such a problem.

On the other hand, similar to knot projections in smooth knot theory, the Legendrian knots admit and are determined by the front projections. By considering the types of the decomposition of the front diagrams, one leads to the notion of normal rulings [3, 13]. In [3], for each $m \geq 0$ as above, it's shown that a weighed count of the (m-graded) normal rulings of the front diagram for Λ, gives a Legendrian isotopy invariant $R^m_\Lambda(z)$, called the m-graded ruling polynomials. It turns out, the ruling polynomials can also be used to distinguish the Chekanov pairs. Ruling polynomials are the analogue of Jones polynomials in smooth knot theory, in the sense that they can also be characterized by skein relations [29, 22].

Moreover, the ruling polynomials also admit a contact geometry interpretation in terms of augmentation numbers:

Proposition 0.0.1 ([15, Thm.1.1]). *The augmentation numbers and the ruling polynomials of Λ determine each other by*

$$\mathrm{aug}(\Lambda, q) = q^{-\frac{d+l}{2}} z^l R^m_\Lambda(z)$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, d is the maximal degree in z of the Laurent polynomial $R^m_\Lambda(z)$, and l is the number of connected components of Λ.

Regarding the structure of the augmentation variety $Aug_m(\Lambda, k)$, the following is known:

Proposition 0.0.2 ([15, Thm.3.4]). *Suppose Λ has the nearly plat front diagram $\pi_{xz}(\Lambda)$ (see Section 1.1), with a fixed Maslov potential μ and l base points such that each connected component has a single base point. Then there's a decomposition of the augmentation variety $\mathrm{Aug}_m(T; k)$ into subvarieties*

$$\mathrm{Aug}_m(T; k) = \sqcup_\rho \mathrm{Aug}^\rho_m(T; k)$$

where ρ runs over all m-graded normal rulings of $\pi_{xz}(\Lambda)$, and

$$\mathrm{Aug}^\rho_m(T; k) \cong (k^*)^{-\chi(\rho)+l} \times k^{r(\rho)}$$

where $\chi(\rho) = c_R - s(\rho)$, c_R *is the number of right cusps in* $\pi_{xz}(\Lambda)$ *and* $s(\rho)$ *is the number of switches of* ρ *(See Definition 2.1.4). Finally,* $r(\rho)$ *is the number of* m*-graded returns if* $m \neq 1$ *and the number of* m*-graded returns and right cusps if* $m = 1$.

Main results

In this article, we will firstly give a tangle approach in the study of Legendrian knots, and generalize the previous results to Legendrian tangles[1]. Legendrian tangles are (special) Legendrian submanifolds in $J^1U \subset \mathbb{R}^3_{x,y,z}$ transverse to the boundary $\partial\overline{J^1U}$, for some open interval U in $\mathbb{R}_x$. Similar to Legendrian knots, one can consider the types of the decompositions of Legendrian tangle fronts. As a generalization, this leads to normal rulings and ruling polynomials for Legendrian tangles. In this case, the boundaries (some set of labeled endpoints) of the tangles are also invariant during a Legendrian isotopy. Hence one can in fact define ruling polynomials $< \rho_L|R^m_T(z)|\rho_R >$ with fixed boundary conditions, for a Legendrian tangle T with a Maslov potential μ (see Section 2.1). Here ρ_L (resp. ρ_R) is a given m-graded normal ruling on the left (resp. right) piece (= parallel strands) T_L (resp. T_R) of T. As the first result, we show the Legendrian invariance and composition axiom for ruling polynomials:

Theorem 0.0.3 (See Theorem 2.1.10)**.** *The* m*-graded ruling polynomials* $< \rho_L|R^m_T(z)|\rho_R >$ *are Legendrian isotopy invariants for* (T, μ).
Moreover, suppose $T = T_1 \circ T_2$ *is the composition of two Legendrian tangles* T_1, T_2*, that is,* $(T_1)_R = (T_2)_L$ *and* $T = T_1 \cup_{(T_1)_R} T_2$*, then the* composition axiom *for ruling polynomials holds:*

$$< \rho_L|R^m_T(z)|\rho_R >= \sum_{\rho_I} < \rho_L|R^m_{T_1}(z)|\rho_I >< \rho_I|R^m_{T_2}(z)|\rho_R >$$

where ρ_I *runs over all the* m*-graded normal rulings of* $(T_1)_R = (T_2)_L$.

On the other hand, generalizing the LCH DGAs for Legendrian knots, one can construct a (bordered) LCH DGA $\mathcal{A}(T, \mu, *_1, \ldots, *_B)$, associated to any Legendrian tangle (T, μ) with base points $*_1, \ldots, *_B$. For example, see [32] and [26, Section.6] in the case when T has the simple front diagram. As usual, one obtains the homotopy invariance of the DGAs. Hence, by a similar procedure as in the case of Legendrian knots, one can consider the associated augmentation varieties $\mathrm{Aug}_m(T, \epsilon_{\rho_L}, \rho_R; k)$ and augmentation numbers $\mathrm{aug}_m(T, \rho_L, \rho_R; q)$, with fixed boundary conditions (ρ_L, ρ_R) as above (see Definition 3.1.10, 3.1.11). These augmentation numbers are again Legendrian isotopy invariants. Moreover, generalizing the previous Proposition 0.0.1 [15, Thm.1.1], we show that:

Theorem 0.0.4 (See Theorem 3.2.7)**.** *Let* T *be a Legendrian tangle equipped with a* $\mathbb{Z}/2r$*-valued Maslov potential* μ *and* B *base points so that each connected component containing*

[1]Throughout the context, Legendrian tangles are assumed to be oriented.

a right cusp has at least one base point. Fix a nonnegative integer m dividing $2r$ and m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively, then the augmentation numbers and ruling polynomials of (T, μ) are related by

$$\mathrm{aug}_m(T, \rho_L, \rho_R; q) = q^{-\frac{d+B}{2}} z^B < \rho_L | R_T^m(z) | \rho_R >$$

where q is the order of a finite field $\mathbb{F}_q$, $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, d is the maximal degree in z of $< \rho_L | R_T^m(z) | \rho_R >$.

Remark 0.0.5. *When T is a Legendrian knot*[2]*, with $B = l$ base points placed on T so that each connected component of T contains a single base point. The left and right pieces of T are empty tangles, hence the boundary conditions become trivial and the theorem reduces to the previous proposition 0.0.1. This gives a new proof of Proposition 0.0.1 [15, Thm.1.1].*

More generally, one can consider the augmentation varieties $\mathrm{Aug}_m(T, \epsilon_{\rho_L}, \rho_R; k)$ with boundary conditions (ϵ_L, ρ_R), where ϵ_L is any m-graded augmentation defining ρ_L of T_L (all such augmentations form an orbit $\mathcal{O}_m(\rho_L; k)$ of the canonical one ϵ_{ρ_L}, see Remark 3.1.7). Similar to Proposition 0.0.2 [15, Thm.3.4], we have the following structure theorem for the augmentation varieties $\mathrm{Aug}_m(T, \epsilon_{\rho_L}, \rho_R; k)$, but with a more natural proof:

Theorem 0.0.6 (See Theorem 3.3.10)**.** *Let (T, μ) be any Legendrian tangle, with B base points placed on T so that* each right cusp is marked*. Fix m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively. Fix $\epsilon_L \in \mathcal{O}_m(\rho_L; k)$. Then there's a decomposition of augmentation varieties into disjoint union of subvarieties*

$$\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k) = \sqcup_\rho \mathrm{Aug}_m^\rho(T, \epsilon_L, \rho_R; k)$$

where ρ runs over all m-graded normal rulings of T such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$. Moreover,

$$\mathrm{Aug}_m^\rho(T, \epsilon_L, \rho_R; k) \cong (k^*)^{-\chi(\rho)+B} \times k^{r(\rho)}.$$

In addition, we pursue a study of the mixed Hodge struture on the (compactly supported) cohomology of the augmentation varieties. For any Legendrian tangle T in the 1-jet bundle $J^1U \hookrightarrow J^1\mathbb{R}_x$, with $U \hookrightarrow \mathbb{R}$ an open interval, the LCH DGAs $\mathcal{A}(T|_V)$ satisfy a co-sheaf/van-Kampen property over open $V \hookrightarrow U$, hence behave like 'fundamental groups'. The invariance of the DGAs $\mathcal{A}(T)$ up to homotopy equivalence ensures we obtain Legendrian isotopy invariants by studying the Hodge theory of their 'representation varieties' (called augmentation varieties). In particular, the study of the augmentation varieties is like that of character varieties, for example, as in [14]. In the case of Legendrian tangles T, the natural objects to consider are augmentation varieties with fixed boundary conditions $\mathrm{Aug}_m(T, \rho_L, \rho_R; k)$, which differ from $\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k)$ only by a specific normalized factor [33]. From the Hodge-theoretic point of view, the previous result (Theorem 3.2.7) concerning

[2] Throughout the context, we make no distinction between 'Legendrian knot' and 'Legendrian link'.

point-counting over finite fields, by [14, Katz's appendix], simply says, the weight (or E-, or virtual Poincaré) polynomials of the augmentation varieties $\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})$ over $\mathbb{C}$, recover the ruling polynomials $< \rho_L | R_T^m(z) | \rho_R >$. Concerning the MHS, the ruling decomposition (Theorem 3.3.10, Remark 3.3.11) naturally induces a spectral sequence converging to the mixed Hodge structure on $\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})$:

Lemma 0.0.7 (See Definition/Proposition 4.1.3, Lemma 4.1.4)**.** *The ruling decomposition for $Aug_m(T, \rho_L, \rho_R; k)$ induces a finite filtration*

$$Aug_m(T, \rho_L, \rho_R; k) = A_D \supset A_{D-1} \supset \ldots \supset A_0 \supset A_{-1} = \emptyset$$

by closed subvarieties, such that, each $A_i - A_{i-1}$ is a disjoint union of the connected components of the form $k^{*\alpha} \times k^{\beta}$. Moreover, it induces a spectral sequence *converging to the compactly supported cohomology of the variety A_D, respecting the mixed Hodge structures (MHS):*

$$E_1^{p,q} = H_c^{p+q}(A_p \setminus A_{p-1}) \Rightarrow H_c^{p+q}(A_D).$$

As some immediate applications, we obtain:

Proposition 0.0.8 (See Proposition 4.2.4, 4.2.5)**.** *The MHS on $H_c^*(\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C}))$ is of Hodge-Tate type.*
Moreover, $H_c^i(\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})) = 0$ for $i < C$, where $C = C(T, \rho_L, \rho_R) := (-\chi(\rho) + B + n'_L) + 2(r(\rho) + A(\rho_L))$ (Remark 3.3.11) is a constant depending only on T, ρ_L, ρ_R. In particular, the cohomology $H_c^*(\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C}))$ vanishes in the lower-half degrees.

In the end, we conclude the article with one example-computation of the MHSs of the augmentation varieties, via this tangle approach.

Organization

In Chapter 1, we review the basic backgrounds in Legendrian knot theory.
In Chapter 2, we give a tangle approach in the study of Legendrian knots. In Section 2.1, we discuss the basics of Legendrian tangles, define the normal rulings and ruling polynomials for Legendrian tangles. Then we prove that the ruling polynomials are Legendrian isotopy invariants and satisfy the composition axiom, the key axiom of a TQFT (Theorem 2.1.10). In Section 2.2, we discuss the LCH DGAs for any Legendrian tangles (not necessarily with nearly plat fronts) via the front projection and resolution construction.
In Chapter 3, we define and study the point-counting of the augmentation varieties. In Section 3.1, we define augmentation varieties and augmentation numbers for Legendrian tangles (with fixed boundary conditions). In Section 3.2, we prove an algorithm to compute the augmentation numbers. The invariance of augmentation numbers and the main Theorem 3.2.7 then follow quickly. The key ingredients of the algorithm are the structures of the augmentation varieties associated to the trivial Legendrian tangle of n parallel strands and

elementary Legendrian tangles. The former is a result about the Barannikov normal forms (Lemma 3.1.5). The latter (Lemma 3.2.3) is dealt with in Section 3.3, where we also prove a stronger result, which leads to a structure theorem (Theorem 3.3.10) for the augmentation varieties associated to any Legendrian tangles.

In Chapter 4, we pursue a study of the mixed Hodge structure on the (compactly supported) cohomology of the augmentation varieties $X = \mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})$. In Section 4.1, we use the ruling decomposition (Theorem 3.3.10) to derive a spectral sequence converging to the mixed Hodge structure on X (Lemma 4.1.4). In Section 4.2, we use the spectral sequence to show that the augmentation variety is of Hodge-Tate type (Proposition 4.2.4), and $H_c^*(X) = 0$ if $* < C$ where C is a constant depending only T and the boundary conditions (ρ_L, ρ_R) (Proposition 4.2.5). In Section 4.2, we give an example-computation for the mixed Hodge structures of the augmentation varieties.

Chapter 1

Background

This chapter collects some background materials concerning Legendrian knot theory.

1.1 Legendrian knot basics

Contact basics

Take the standard contact three-space $\mathbb{R}^3_{x,y,z} = J^1(\mathbb{R}_x) = T^*\mathbb{R}_x \times \mathbb{R}_z$ with contact form $\alpha = dz - ydx$. The Reeb vector field of α is then $R_\alpha = \partial_z$. We consider a (one-dimensional) Legendrian submanifold (termed as *knot* or *link*) Λ in this three space $\mathbb{R}^3$. The *front* and *Lagrangian projections* of Λ are $\pi_{xz}(\Lambda)$ and $\pi_{xy}(\Lambda)$ respectively, with the obvious projections $\pi_{xz} : \mathbb{R}^3_{x,y,z} \to \mathbb{R}^2_{x,z}$ and $\pi_{xy} : \mathbb{R}^3_{x,y,z} \to \mathbb{R}^2_{x,y}$.

Front diagrams

We will always *assume* the Legendrian link $\Lambda \subset \mathbb{R}^3$ is in a generic position inside its Legendrian isotopy class. So, the front projection $\pi_{xz}(\Lambda)$ gives a *front diagram* (i.e. an immersion of a finite union of circles into $\mathbb{R}^2_{xz}$ away from finitely many points (cusps) having no vertical tangent, which is also an embedding away from finitely many points (cusps and transversal crossings)). The significance of front diagrams is that, any Legendrian link is uniquely determined by its front projection. That is, the y-coordinate can be recovered from the x and z-coordinates of the front projection as the slope, via the Legendrian condition $dz - ydx = 0 \Rightarrow y = dz/dx$. In other words, in passing to the front projection, we loss no information. Note also that, near each crossing of a front diagram, the strand of the lesser slope is always the over-strand.

Given a front diagram, the *strands* of $\pi_{xz}(\Lambda)$ are the maximally immersed connected submanifolds, the *arcs* of $\pi_{xz}(\Lambda)$ are the maximally embedded connected submanifolds and the *regions* are the maximal connected components of the complement of $\pi_{xz}(\Lambda)$ in $\mathbb{R}^2_{xz}$.

We say a front diagram in $\mathbb{R}^2_{x,z}$ is *plat* if the crossings have distinct x-coordinates, all the left cusps have the same x-coordinate and likewise for the right cusps. We say a front diagram is *nearly plat*, if it's a perturbation of a plat front diagram, so that the crossings and cusps all have different x-coordinates. We can always make the front diagram $\pi_{xz}(\Lambda)$ (nearly) plat by smooth isotopies and Legendrian Reidemeister II moves (see FIGURE 1.1.2).

We say a front diagram in $\mathbb{R}^2_{xz}$ is *simple* if it's smooth isotopic to a front whose right cusps have the same x-coordinate. For example, any (nearly) plat front diagram is simple.

Resolution construction

In this article, we will use both the front and Lagrangian projections. Hence, it's often necessary to translate between the 2 projections in some simple way. This can be realized by the resolution construction [24, Prop.2.2]. Given the front diagram $\pi_{xz}(\Lambda)$, we can obtain the Lagrangian projection $\pi_{xy}(\Lambda')$ of a link Λ' Legendrian isotopic to Λ, via a resolution procedure as in FIGURE 1.1.1. We say that Λ' is obtained from Λ by *resolution construction*. Note that the same conclusion applies to Legendrian tangles (see Section 2.1 for the definition.)

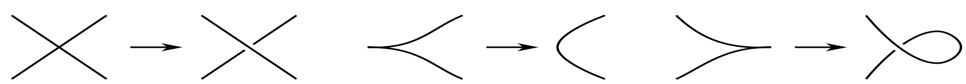


Figure 1.1.1: Resolving a front into the Lagrangian projection of a Legendrian isotopic link/tangle.

Legendrian Reidemeister moves

It's well known that any smooth knot can be represented by a knot diagram, and any 2 knot diagrams represent smoothly isotopic knots if and only if they differ by smooth isotopy and a finite sequence of 3 types of topological Reidemeister moves. There's an analogue for Legendrian knots via front diagrams. That is, 2 front diagrams in $\mathbb{R}^2_{xz}$ represent the same Legendrian isotopy class of Legendrian knots in $\mathbb{R}^3_{x,y,z}$ if and only if they differ by a finite sequence of smooth isotopies and the following *Legendrian Reidemeister moves* of 3 types ([34]):

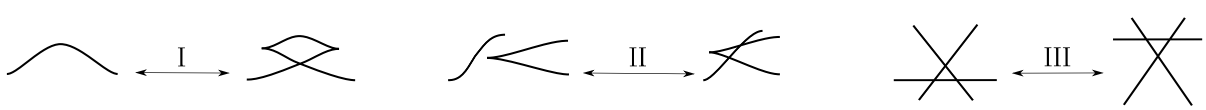


Figure 1.1.2: The 3 types of Legendrian Reidemeister moves relating Legendrian-isotopic fronts. Reflections of these moves along the coordinate axes are also allowed.

Topological knot type

The *topological knot type* of a Legendrian knot is the smooth isotopy class of its underlying smooth knot. Clearly, this defines a Legendrian isotopy invariant. As a consequence, all topological knot invariants ([16, 12, 35] as well as their "categorified" versions ([17, 18, 19]) are automatically Legendrian isotopy invariants.

Thurston-Bennequin number

Given an oriented Legendrian knot Λ in $(\mathbb{R}^3_{x,y,z}, \alpha = dz - ydx)$, the *Thurston-Bennequin number* (denoted by $tb(\Lambda)$) measures the twisting of the oriented contact plane field along the knot. It can be defined as the linking number of the Legendrian knot and its push-off along the Reeb direction $R_\alpha = \partial_z$, that is, $tb(\Lambda) := lk(\Lambda, \Lambda + \epsilon z)$. The geometric definition makes it automatically a Legendrian invariant. On the other hand, project the Legendrian knot down to the front plane $\mathbb{R}^2_{xz}$, the number can be computed via the front diagram: $tb(\Lambda) = wr(\pi_{xz}(\Lambda)) - c(\pi_{xz}(\Lambda))$, where wr is the writhe number and c is the number of right cusps. It's then also easy to check the Legendrian isotopy invariance via Legendrian Reidemeister moves.

Rotation number

Given an oriented connected Legendrian knot Λ in $\mathbb{R}^3$, the *rotation number* $r(\Lambda)$ is the obstruction to extending the tangent vector field of Λ to a nonzero section of the contact plane field over a Seifert surface (an embedded compact oriented surface) bounding Λ. It can also be computed via the front diagram: $r(L) = 1/2(U(\pi_{xz}(\Lambda) - D(\pi_{xz}(\Lambda))))$, where U (resp. D) is the number of *up (resp. down) cusps* (= a cusp near which the orientation of $\pi_{xz}(\Lambda)$ goes up (resp. down)). This is another example where the Legendrian isotopy invariance can be checked via Legendrian Reidemeister moves. By definition, the rotation number depends via a sign on the orientation. For an oriented multi-component Legendrian knot Λ, we usually define its rotation number $r(\Lambda)$ as the *gcd* of the rotation numbers of its components.

Maslov potential

Given a Legendrian knot Λ with front diagram $\pi_{xz}(\Lambda)$. Let $r = |r(\Lambda)|$ and n be a nonnegative integer. A $\mathbb{Z}/n\mathbb{Z}$-valued *Maslov potential* of $\pi_{xz}(\Lambda)$ is a map

$$\mu : \{\text{strands of } \pi_{xz}(\Lambda)\} \to \mathbb{Z}/n\mathbb{Z}$$

such that near any cusp, have $\mu(\text{upper strand}) = \mu(\text{lower strand}) + 1$. Such a Maslov potential exists if and only if $2r$ is a multiple of n. In particular, the existence of a $\mathbb{Z}$-valued Maslov potential implies that every component of Λ has rotation number 0. We will often fix for Λ a $\mathbb{Z}/2r$-valued Maslov potential μ.

We usually view the topological knot type, the Thurston-Bennequin number and the rotation number as the *classical invariants* of (oriented) Legendrian knots. It turns out, they are not sufficient to classify Legendrian knots up to Legendrian isotopy. In [4], a pair of Legendrian knots (they represent the same class 5_2 in the classification of topological knots) having the same classical invariants, were shown to be distinguished by a new Legendrian isotopy invariants, the LCH DGA (or Chekanov-Eliashberg DGA) (See Section 1.2 below).

1.2 LCH differential graded algebras

LCH DGA via Lagrangian projection

Here we recall the Legendrian contact homology differential graded algebra for Legendrian links in $\mathbb{R}^3$ [11]. The version of DGAs we need will also allow an arbitrary number of base points placed on the Legendrian links [25, 26]. The construction is naturally formulated via Lagrangian projection.

Initial data: Let Λ be an oriented Legendrian link in $\mathbb{R}^3_{x,y,z}$, with rotation number $r(\Lambda)$. Take $r = |r(\Lambda)|$ and fix a $\mathbb{Z}/2r$-valued Maslov potential μ of $\pi_{xz}(\Lambda)$. Let $*_1, \ldots, *_B$ be the base points placed on Λ, avoiding the crossings of the Lagrangian projection $\pi_{xy}(\Lambda)$, such that each component of Λ contains at least one base point. Denote by $\{a_1, \ldots, a_R\}$ the set of crossings of $\pi_{xy}(\Lambda)$, corresponding to the Reeb chords of the Reeb vector field $R_\alpha = \partial_z$.

The $\mathbb{Z}/2r$-graded LCH DGA $\mathcal{A} = \mathcal{A}(\Lambda, \mu, *_1, \ldots, *_B)$ is then defined as follows:

As an algebra: $\mathcal{A}$ is the associative unital algebra $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_B^{\pm 1}] < a_1, \ldots, a_R >$ over the commutative ring $\mathbb{Z}[t_i^{\pm 1}, 1 \leq i \leq B]$, freely generated by $a_j, 1 \leq j \leq R$. The generators t_i, t_i^{-1} can be regarded as information encoded at the base point $*_i$.

The grading: The algebra $\mathcal{A}$ is assigned a $\mathbb{Z}/2r$-grading via $|x \cdot y| = |x| + |y|, |t_i| = |t_i^{-1}| = 0$ and $|a_j|$ is defined as follows. The 2 endpoints of the Reeb chord a_j belong to 2 distinct strands of the front projection $\pi_{xz}(\Lambda)$. Near the upper (lower) endpoint of a_j, the over-strand (understrand) can be parameterized as $x \to (x, z = f_u(x))$(resp. $(x, z = f_l(x))$), and $f_u'(x(a_j)) = f_l'(x(a_j))$. By the generic assumption of Λ, $f_u - f_l$ attains either a local maximum (or minimum) at a_j, accordingly we define $|a_j| = \mu(\text{over-strand}) - \mu(\text{under-strand})+$ either 0(or-1).

The differential ∂: To define the differential ∂ on $\mathcal{A}$, we firstly impose the *Leibniz Rule* $\partial(x \cdot y) = (\partial x) \cdot y + (-1)^{|x|} \cdot \partial y$ and $\partial(t_i) = \partial(t_i^{-1}) = 0$. It then suffices to define the differential ∂a_j for each Reeb chord. Intuitively, ∂a_j is a weighted count of boundary-punctured holomorphic disks in the symplectization $(\mathbb{R}_\tau \times \mathbb{R}^3_{x,y,z}, d(e^\tau \alpha)$ with boundary along the Lagrangian $\mathbb{R}_\tau \times \Lambda$, with one positive boundary puncture limiting to the Reeb chord a_j at $+\infty$, and several negative boundary punctures limiting to some Reeb chords at $-\infty$.

In our case, we can describe the differential combinatorially. Given Reeb chords $a = a_j$ and $b_1, \ldots, b_n$ for some $n \geq 0$. Let $D_n^2 = D^2 - \{p, q_1, \ldots, q_n\}$ be a fixed oriented disk with $n+1$ boundary punctures $p, q_1, \ldots, q_n$, arranged in a counterclockwise order.

Definition 1.2.1 (Admissible disks)**.** *Define the moduli space* $\Delta(a; b_1, \dots, b_n)$ *to be the space of* admissible disks u *of* $\pi_{xy}(\Lambda)$ *up to re-parametrization, that is,*

- $u : (D_n^2, \partial D_n^2) \to (\mathbb{R}^2_{x,y}, \pi_{xy}(\Lambda))$ *is a smooth orientation-preserving immersion, extends continuously to* D^2*;*
- $u(p) = a, u(q_i) = b_i (1 \leq i \leq n)$ *and* u *sends a neighborhood of* p *(resp.* q_i*) in* D^2 *to a single quadrant of* a *(resp.* b_i*) with positive (resp. negative)* Reeb sign *(see below).*

Reeb signs: *Near a crossing of the Lagrangian projection* $\pi_{xy}(\Lambda)$*, the 2 quadrants lying in the counterclockwise (resp. clockwise) direction of the over-strand are assigned* positive *(resp.* negative*)* Reeb signs*. See Figure 1.2.1 (left).*

For each $u \in \Delta(a; b_1, \dots, b_n)$, walk along $u(\partial D^2)$ starting from a, we encounter a sequence $s_1, \dots, s_N (N \geq n)$ of crossings (excluding a) and base points of $\pi_{xy}(\Lambda)$. We then define the weight $w(u)$ of u as follows

Definition 1.2.2. $w(u) := s(u)w(s_1)\dots w(s_N)$*, where*

(i) $w(s_i) = b_j$ *if* s_i *is the crossing* b_j*.*

(ii) $w(s_i) = t_j$*(resp.* t_j^{-1}*) if* s_i *is the base point* $*_j$*, and the boundary orientation of* $u(\partial D^2)$ *agrees (resp. disagrees) with the orientation of* Λ *near* $*_j$*.*

(iii) $s(u)$ *is the product of the* orientation signs *(see below) of the quadrants near* a *and* $b_1, \dots, b_n$ *occupied by* u*.*

Orientation signs: *We will use the same convention as [26]. That is, at each crossing* a *such that* $|a|$ *is even, we assign negative* orientation signs *to the 2 quadrants that lie on any chosen side of the under-strand of* a*; We assign positive* orientation signs *to all the other quadrants. See Figure 1.2.1 (right).*

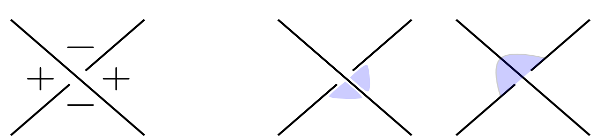


Figure 1.2.1: Left: the Reeb signs of the quadrants at a crossing in a Lagrangrian projection. Right: the two possible choices of orientation signs for the quadrants at a crossing of even degree in a Lagrangian projection. The shaded quadrants have negative orientation signs and the unshaded quadrants have positive orientation signs. At a crossing of odd degree, all the four quadrants have positive orientation signs.

Now we can define the differential of $a = a_j$:

$$\partial a = \sum_{n,b_1,\ldots,b_n} \sum_{u \in \Delta(a;b_1,\ldots,b_n)} w(u) \tag{1.2.0.1}$$

Theorem 1.2.3 ([4, 11])**.** *$(\mathcal{A}, \partial)$ is a $\mathbb{Z}/2r$-graded DGA with* $\deg(\partial) = -1$.

Invariance of LCH DGA

We have seen that, the definition of the LCH DGA $\mathcal{A}(\Lambda)$ associated to a Legendrian link Λ depends on several choices: a specific choice of the representative of Λ inside its Legendrian isotopy class, and a choice of base points. Here we review that the LCH DGA $\mathcal{A}(\Lambda)$ is a Legendrian isotopy invariant, up to a *stable isomorphism*, in particular, up to homotopy equivalence of $\mathbb{Z}/2r$-graded DGAs.

Definition 1.2.4. *An* (algebraic) stabilization *of a $\mathbb{Z}/2r$-graded DGA $(\mathcal{A}, \partial)$ is a $\mathbb{Z}/2r$-graded DGA $(S(\mathcal{A}), \partial')$ obtained by adding 2 new free generators e and f, with $|e| = |f| + 1$, such that $\partial'|_{\mathcal{A}} = \partial$ and $\partial' e = f, \partial' f = 0$. Two $\mathbb{Z}/2r$-graded DGAs $(\mathcal{A}, \partial)$ and $(\mathcal{A}', \partial')$ are* stable isomorphic*, if they are isomorphic as $\mathbb{Z}/2r$-graded DGAs, after possibly stabilizing each finitely many times.*

Theorem 1.2.5 ([25, Thm.2.20],[11, Thm.3.10])**.** *The isomorphism class of $(\mathcal{A}(\Lambda), \partial)$ as an DGA is independent of the locations of the base points on each connected component of Λ. The stable isomorphism class of $(\mathcal{A}(\Lambda), \partial)$ is invariant under Legendrian isotopy of Λ.*

LCH DGA via front projection

Assume the front projection Λ is simple (see Section 1.1 for the definition). Then the LCH DGA also admits a simple front projection description.

The resolution construction of Λ gives a Legendrian isotopic link $\Lambda' = \mathrm{Res}(\Lambda)$, whose Reeb chords are in one-to-one correspondence with the crossings and right cusps of $\pi_{xz}(\Lambda)$. We will denote by $\mathcal{A}(\Lambda_{xz}, \mu, *_1, \ldots *_B; k)$ the LCH DGA associated to $\mathrm{Res}(\Lambda)$. Denote by $\{a_1, \ldots a_p\}$ (resp. $\{c_1, \ldots c_q\}$) the set of crossings (resp., of right cusps) of $\pi_{xz}(\Lambda)$. Under the correspondence, the algebra is generated over $\mathbb{Z}[t_i^{\pm 1}, 1 \leq i \leq B]$ by $\{a_k, 1 \leq k \leq p, c_k, 1 \leq k \leq q\}$. The grading is given by: $|t_k^{\pm}| = 0, |a_k| = \mu(\text{over-strand}) - \mu(\text{under-strand})$ and $|c_k| = 1$. One can also translate the definition of the differential for $\mathrm{Res}(\Lambda)$ into the front projection $\pi_{xz}(\Lambda)$. The definition uses the same formula by "counting" the disks in $\pi_{xz}(\Lambda)$ plus the additional "invisible disks", one for each right cusp. An "invisible disk" (See FIGURE 1.1.1, the last picture) corresponds to a disk with one unique corner on its left at the crossing of $\pi_{xy}(\mathrm{Res}(\Lambda))$ corresponding to the right cusp of $\pi_{xz}(\Lambda)$.

Chapter 2

A tangle approach for studying Legendrian knots

In this chapter, we give a tangle approach in the study of Legendrian knots. More precisely, we generalize the basic concepts and results of Legendrian knots to Legendrian tangles, including ruling polynomials and LCH DGAs. We also show that the ruling polynomials satisfy the composition axiom, illustrating that they're the Legendrian analogues of the Jones polynomials in topological knot theory.

2.1 Ruling polynomials for Legendrian tangles

Legendrian tangles

Fix $U = (x_L, x_R)$ to be a open interval in $\mathbb{R}_x$ ($-\infty \leq x_L < x_R \leq \infty$), so the standard contact form $\alpha = dz - ydx$ induces a standard contact structure on $J^1U = U \times \mathbb{R}^2_{y,z}$. A *Legendrian tangle* T is a Legendrian submanifold in J^1U transverse to the boundary $\partial J^1(\overline{U})$. Typical examples of Legendrian tangles can be obtained from a Legendrian link front by removing the parts outside of a vertical strip in $\mathbb{R}^2_{xz}$.

Remark 2.1.1. *In Section 1.1, notice that the same concepts (*front diagrams, strands, arcs, regions, etc.*) can be introduced for any Legendrian tangle T in J^1U. We only have to replace Λ by T and $\mathbb{R}^2_{xz}$ by $U \times \mathbb{R}_z$ there. Similarly, in Section 1.1, the same procedure applies to define* Maslov potentials *for T.*

As usual, we will assume T has a generic front projection.[1] We equip T with a $\mathbb{Z}/2r$-valued Maslov potential μ for some fixed $r \geq 0$. Denote by n_L (resp. n_R) the number of left (resp. right) end-points on T.

[1]From now on, we will make no distinction between the Legendrian tangle T and its front projection $\pi_{xz}(T)$.

We say 2 Legendrian tangles in J^1U are *Legendrian isotopic* if there's an isotopy between them along Legendrian tangles in J^1U. Note that during the Legendrian isotopy, we require the *ordering* via z-coordinates of the end-points is preserved. That is, for two (say, left) end-points p_1, p_2, they necessarily have the common x-coordinate x_L, take any path γ in $\partial J^1(\overline{U})$ from p_2 to p_1, then we say $p_1 > p_2$ if $z(p_1) - z(p_2) = \int_\gamma \alpha > 0$. Then, similar to the case of Legendrian links, two (generic) Legendrian tangle fronts are Legendrian isotopic *if and only if* they differ by a finite sequence of smooth isotopies (preserving the ordering of the end-points) and *Legendrian Reidemeister moves* of the 3 types (see Figure 1.1.2).

Normal Rulings and Ruling polynomials

Similar to Legendrian knots, we can introduce the notion of m-graded normal rulings and Ruling polynomials for any Legendrian tangles. Given a Legendrian tangle T, with $\mathbb{Z}/2r$-valued Maslov potential μ for some fixed $r \geq 0$. Fix a nonnegative integer m dividing $2r$.

Assume that the numbers n_L, n_R of the left endpoints and right endpoints of T are both *even*. For example, any Legendrian tangle obtained from cutting a Legendrian link front along 2 vertical lines, satisfies this assumption.

Recall that in Remark 2.1.1, we have introduced the notions of *arcs*, *crossings*, *cusps* and *regions* of the front $\pi_{xz}(T)$. In particular, the front diagram is divided into arcs, crossings and cusps. For example, an arc begins at a cusp, a crossing or an end-point, going from left to right, and ends at another cusp, crossing or end-point, meeting no cusp or crossing in-between. Given a crossing a of the front T, its degree is given by $|a| := \mu(\text{over-strand}) - \mu(\text{under-strand})$.

Definition 2.1.2. *We say an embedded (closed) disk of $\overline{U} \times \mathbb{R}_z$, is an* eye *of the front T, if it is the union of (the closures of) some* regions*, such that the boundary of the disk in $U \times \mathbb{R}_z$, being the union of arcs, crossings and cusps, consists of 2* paths*, starting at the same left cusp or a pair of left end-points, going from left to right through arcs and crossings, meeting no cusps in-between, and ending at the same right cusp or a pair of right end-points.*

Definition 2.1.3. *A m-*graded normal Ruling *ρ of (T, μ) is a* partition *of the set of arcs of the front T into the boundaries in $U \times \mathbb{R}_z$ of* eyes *(say $e_1, \ldots, e_n$), or in other words,*

$$\sqcup \text{ arcs of } T = \sqcup_{i=1}^{n} (\partial e_i \setminus \{\text{crossings, cusps}\}) \cap U \times \mathbb{R}_z,$$

and such that the following conditions are satisfied:

(1). If some eye e_i starts at a pair of left end-points (resp. ends at a pair of right end-points), we require $\mu(\text{upper-end-point}) = \mu(\text{lower-end-point}) + 1 (\text{mod} m)$.

(2). Call a crossing a a switch*, if it's contained in the boundary of some eye e_i. In this case, we require $|a| = 0 (\text{mod} m)$.*

(3). Each switch a is clearly contained in exactly 2 eyes, say e_i, e_j. We require the relative positions of e_i, e_j near a to be in one of the 3 situations in Figure 2.1.1(top row).

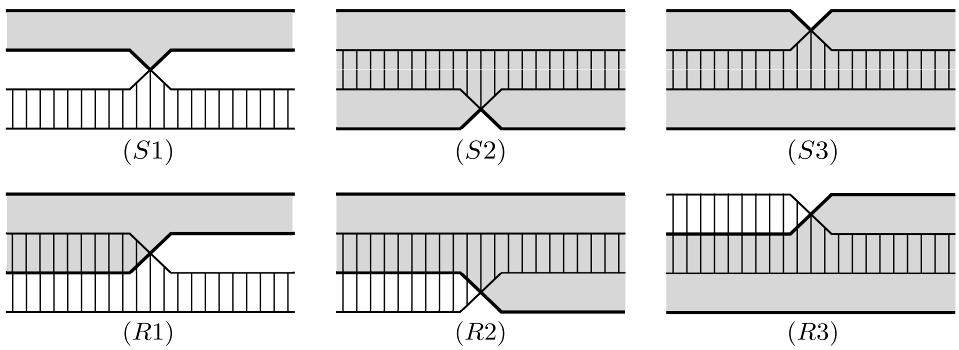


Figure 2.1.1: Top row: The behavior (of the 2 eyes e_i, e_j) of a m-graded normal ruling ρ at a *switch* (where e_i and e_j are the dashed and shadowed regions respectively), where the crossings are required to have degree 0 (mod$m$). Bottom row: The behavior (of the 2 eyes e_i, e_j) of ρ at a return. Three more figures omitted: The 3 types of departures obtained by reflecting each of $(R1)$-$(R3)$ with respect to a vertical axis.

Definition 2.1.4. *Given a Legendrian tangle (T, μ), let ρ be a m-graded normal ruling of (T, μ), and let a be a crossing. Then, a is called a* return *if the behavior of ρ at a is as in Figure 2.1.1(bottom row). a is called a* departure *if the behavior of ρ at a looks like one of the three pictures obtained by reflecting each of $(R1) - (R3)$ in Figure 2.1.1(bottom row) with respect to a vertical axis. Moreover, returns (resp. departures) of degree 0 modulo m are called m-*graded returns *(resp. m-*graded departures*) of ρ.*
Define $s(\rho)$ *(resp. $d(\rho)$) to be the number of switches (resp. m-graded departures) of ρ.*
Define $r(\rho)$ *to be the number of m-graded returns of ρ if $m \neq 1$, and the number of m-graded returns and right cusps if $m = 1$.*

Remark 2.1.5. *If we fix the pairing ρ_L of left end-points, a m-graded normal Ruling ρ determines and is determined by a subset, denoted by the same symbol ρ, of the switches in the set of crossings of T. In this case, we will usually make no distinction between a m-graded normal Ruling and its set of switches.*

Definition 2.1.6. *Given a m-graded normal Ruling ρ of a Legendrian tangle (T, μ), denote by $e_1, \ldots, e_n$ the eyes in $J^1(\overline{U})$ defined by ρ. The* filling surface *S_ρ of ρ is the the disjoint union $\sqcup_{i=1}^{n} e_i$ of the eyes, glued along the switches via half-twisted strips. This is a compact surface possibly with boundary. See FIGURE 2.1.2 for an example.*

Let T_L (resp. T_R) be the left (resp. right) pieces T near the left (resp. right) boundary. It's clear that any m-graded normal Ruling ρ of T restricts to a m-graded normal Ruling

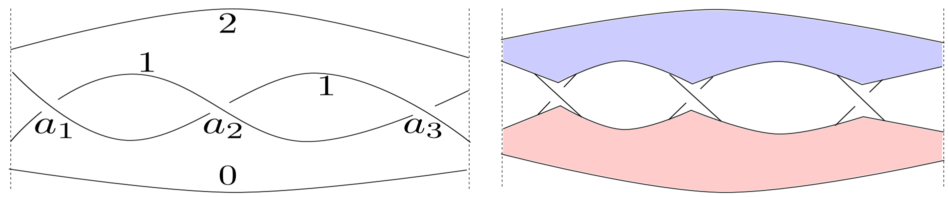


Figure 2.1.2: Left: a Legendrian tangle front T with 3 crossings a_1, a_2, a_3, the numbers indicate the values of the Maslov potential μ on each of the 4 strands. Right: the filling surface for a normal ruling of T by gluing the 2 eyes along the 3 switches via half-twisted strips.

of the left piece T_L (resp. of the right piece T_R), denoted by $r_L(\rho)$ *or* $\rho|_{T_L}$ *(resp.* $r_R(\rho)$ *or* $\rho|_{T_R}$*).*

Definition 2.1.7. *Fix a m-graded normal Ruling ρ_L (resp. ρ_R) of T_L (resp. T_R). We define a Laurent polynomial $<\rho_L|R^m_{T,\mu}(z)|\rho_R>=<\rho_L|R^m_T(z)|\rho_R>$ in $\mathbb{Z}[z,z^{-1}]$ by*

$$<\rho_L|R^m_T(z)|\rho_R>:=\sum_{\rho:\ r_L(\rho)=\rho_L, r_R(\rho)=\rho_R} z^{-\chi(\rho)} \tag{2.1.0.1}$$

where the sum is over all m-graded normal Rulings ρ such that $r_L(\rho)=\rho_L, r_R(\rho)=\rho_R$. $\chi(\rho)$ is called the Euler characteristic *of ρ and defined by*

$$\chi(\rho) := \chi(S_\rho) - \chi(S_\rho|_{x=x_R}). \tag{2.1.0.2}$$

where x_R is the right endpoint of the open interval $U=(x_L,x_R)$ and $\chi(S_\rho)$ (resp. $\chi(S_\rho|_{x=x_R})$) is the usual Euler characteristic of S_ρ (resp. $S_\rho|_{x=x_R}$). Equivalently, $\chi(\rho)=\chi_c(S_\rho|_{x_L\leq x<x_R})$ is the Euler characteristic with compact support of $S_\rho|_{x_L\leq x<x_R}$. Also, notice that when $x_R=\infty$, $S_\rho|_{x=x_R}$ is empty with vanishing Euler characteristic.

*We will call $<\rho_L|R^m_T(z)|\rho_R>$ the m-*graded Ruling polynomial *of T with boundary conditions (ρ_L,ρ_R).*

Remark 2.1.8. *Given a m-graded normal Ruling ρ, with $n_L=2n'_L$ (resp. $n_R=2n'_R$) left (resp. right) end-points and c_L (resp. c_R) left (resp. right) cusps, then $S_\rho|_{x=x_R}$ is the disjoint union of n'_R closed line segments and $n=n'_L+c_L=n'_R+c_R$ is the number of eyes in ρ. Hence, $\chi(S_\rho|_{x=x_R})=n'_R$ is independent of ρ and we get a simple computation formula*

$$\chi(\rho)=c_R-s(\rho) \tag{2.1.0.3}$$

where $s(\rho)$ is defined in Definition 2.1.4. In particular, when T is a Legendrian link, the definition here coincides with the usual definition [15] of Ruling polynomials for Legendrian links.

Moreover, when T is a trivial Legendrian tangle of $2n$ parallel strands, then

$$< \rho_L | R_T^m(z) | \rho_R >= \delta_{\rho_L, \rho_R}.$$

This may be called the Identity axiom *for Ruling polynomials, see Remark 2.1.11 below.*

Invariance and composition axiom

Given a Legendrian tangle T, let's denote by NR_T^m (resp. $\mathrm{NR}_T^m(\rho_L, \rho_R)$) the set of m-graded normal Rulings of T (resp. those with boundary conditions (ρ_L, ρ_R)).

Lemma 2.1.9. *Given a Legendrian isotopy h between 2 Legendrian tangles T, T', preserving the Maslov potentials μ, μ', there's a canonical bijection between the set of m-graded normal Rulings of T and T'*

$$\phi_h : \mathrm{NR}_T^m \xrightarrow{\sim} \mathrm{NR}_{T'}^m$$

commuting with the restrictions r_L, r_R, and such that for any m-graded normal Ruling ρ, S_ρ and $\phi(S_\rho)$ are homeomorphic, relative to the boundary pieces at $x = x_L$ and $x = x_R$.

Note that for such 2 Legendrian isotopic tangles $(T, \mu), (T', \mu')$, their left and right pieces are necessarily identical: $T_L = T'_L, T_R = T'_R$.

As a consequence of Lemma 2.1.9, we obtain

Theorem 2.1.10. *The m-graded Ruling polynomials $< \rho_L | R_T^m(z) | \rho_R >$ are Legendrian isotopy invariants for (T, μ).*
Moreover, suppose $T = T_1 \circ T_2$ is the composition of two Legendrian tangles T_1, T_2, that is, $(T_1)_R = (T_2)_L$ and $T = T_1 \cup_{(T_1)_R} T_2$, then the composition axiom *for Ruling polynomials holds:*

$$< \rho_L | R_T^m(z) | \rho_R >= \sum_{\rho_I} < \rho_L | R_{T_1}^m(z) | \rho_I >< \rho_I | R_{T_2}^m(z) | \rho_R > \tag{2.1.0.4}$$

where ρ_I runs over all the m-graded normal rulings of $(T_1)_R = (T_2)_L$.

Proof. The invariance of Ruling polynomials follows immediately from Lemma 2.1.9. As for the composition axiom, let ρ be any m-graded normal ruling of T such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$. Let T_1, T_2 be Legendrian tangles over the open intervals $(x_L, x_I), (x_I, x_R)$ respectively. Take $\rho_I = \rho|_{(T_1)_R}, \rho_1 = \rho|_{T_1}, \rho_2 = \rho|_{T_2}$. Let $S_\rho, S_{\rho_1}, S_{\rho_2}$ be the filling surfaces of ρ, ρ_1, ρ_2 over T, T_1, T_2 respectively. Then $S_{\rho_1}|_{x=x_I} = S_{\rho_2}|_{x=x_I}$ and $S_\rho = S_{\rho_1} \cup_{S_{\rho_1}|_{x=x_I}} S_{\rho_2}$, it follows that $\chi(\rho) = \chi(S_\rho) - \chi(S_\rho|_{x=x_R}) = \chi(S_{\rho_1}) - \chi(S_{\rho_1}|_{x=x_I}) + \chi(S_{\rho_2}) - \chi(S_{\rho_2}|_{x=x_R}) = \chi(\rho_1) + \chi(\rho_2)$. Now, the composition axiom follows immediately from applying this into Definition 2.1.7 of Ruling polynomials. □

Remark 2.1.11. *The previous theorem suggests a "TQFT" interpretation of Ruling polynomials for Legendrian tangles, strengthen the analogue that Ruling polynomials are Legendrian versions of Jones polynomials, which fits into a TQFT in smooth knot theory.*

Morally, we may regard Legendrian tangles (T, μ) as 1-dimensional cobordisms from the 0*-manifold of left endpoints with additional structures (equivalently, T_L) to the* 0*-manifold of right endpoints with additional structures (equivalently, T_R). In other words, the* 0*-manifolds with additional structures (equivalently, trivial Legendrian tangles (E, μ_0) of even parallel strands) and* 1*-dimensional cobordisms (equivalently, Legendrian tangles (T, μ)) form a special 1-dimensional cobordism category $\mathcal{LT}_1$. Now, we can view Ruling polynomials $R^m_T(z)$ as a "1-dimensional TQFT" functor R^m from $\mathcal{LT}_1$ into the category of free modules of finite rank over $K := \mathbb{Z}[z, z^{-1}]$ (See [1] for the basic concepts of TQFTs).*

More precisely, associate to any any trivial Legendrian tangle (E, μ_0) of even parallel strands (viewed as an object of $\mathcal{LC}_1$), R^m assigns the free module $R^m(E)$ over K generated by all the m-graded normal rulings of E; Associate to any 1-dimensional cobordism (T, μ), R^m assigns the K-module morphism $R^m(T) := R^m_T(z)$ from $R^m(T_L)$ to $R^m(T_R)$, defined by the matrix coefficients $< \rho_L | R^m_T(z) | \rho_R >$. The previous theorem and Remark 2.1.8 shows that R^m is indeed a functor.

Proof of Lemma 2.1.9. As any Legendrian isotopy of Legendrian tangles is a composition of a finite sequence of smooth isotopies and Legendrian Reidemeister moves of the 3 types (see FIGURE 1.1.2), it suffices to show the proposition for a single smooth isotopy or each of the 3 types of Legendrian Reidemeister moves. As always, we assume the x-coordinates of the crossings and cusps of the tangle fronts in question are all distinct. The proof is essentially done for each case by pictures.

If h is a smooth isotopy. This case is actually not trivial, as the ordering by x-coordinates of the crossings and cusps will change during a smooth isotopy, which will affect the set of switches of the normal Rulings. We illustrate only one such a case (FIGURE 2.1.3), the other cases are either similar or trivial. Let a, b be 2 neighboring crossings of T, say $x(a) < x(b)$, and the smooth isotopy h moves a to the right of b (i.e. $x(a) > x(b)$ after the isotopy), with the remaining part fixed.

Given a m-graded normal ruling ρ of T before the smooth isotopy, if ρ has no switches at a, b, take $\phi_h(\rho)$ to be the obvious m-graded normal ruling corresponding to ρ. In particular, $\phi_h(\rho)$ has no switches at a, b either. It's also clear that the filling surfaces S_ρ and $S_{\phi_h(\rho)}$ are homeomorphic relative to the boundary pieces at $x = x_L$ and $x = x_R$.

If ρ has a single switch at a or b, say b, then $|b| = 0(\text{mod} m)$. The switch b belongs to 2 eyes of ρ, say e_1, e_2. Recall that each eye has 2 paths (see definition 2.1.2) on the boundary going from left to right, let's call them the *upper-path* and *lower-path* according to their z-coordinates. In our case, each of the 2 eyes e_1, e_2 has one path containing b. If the the remaining 2 companion paths of e_1, e_2 contain at most one of the two strands at the crossing a, again $\phi_h(\rho)$ is taken to be the obvious normal ruling corresponding to ρ with a switch at b, no switch at a. The proposition holds trivially. If the remaining 2 companion paths of e_1, e_2 restrict to the strands near a. By the first condition in the definition 2.1.3 of a m-graded normal ruling, we also have that $|a| = 0(\text{mod} m)$). We look at the relative positions of the 2 eyes e_1, e_2 in the vertical strip near a, b. We consider only one situation illustrated by FIGURE 2.1.3, the others are entirely similar. In this case, the picture on the

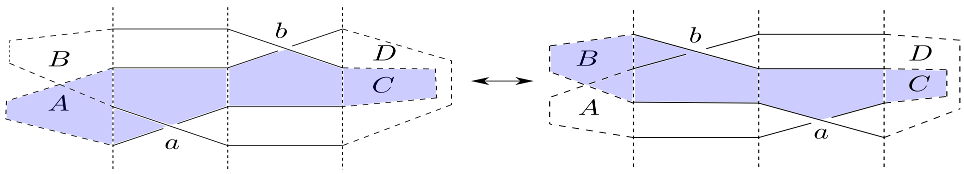


Figure 2.1.3: The bijection between m-graded normal Rulings under smooth isotopy: The left hand side is a m-graded normal ruling ρ with a switch at b, instead the corresponding m-graded normal ruling $\phi_h(\rho)$ on the right hand side has a switch at a. A, B, C, D are the parts of the filling surface S_ρ (as well as $S_{\phi_h(\rho)}$) outside the vertical strip drawn in the picture.

left gives ρ near a, b. We can define $\phi_h(\rho)$ to be the m-graded normal ruling (the picture on the right) having a switch at a, no switch at b and with the same remaining part as ρ. It's easy to see from FIGURE 2.1.3 that $\phi_h(\rho)$ satisfies the conditions in the definition 2.1.3. Moreover, denote the remaining parts of the filling surface S_ρ by A, B, C, D respectively as in FIGURE 2.1.3, then S_ρ is: glue A, C by a strip, call the result $A \leftrightarrow C$, glue B, D by a strip, get $B \leftrightarrow D$, then glue $A \leftrightarrow C$ and $B \leftrightarrow D$ by a half-twisted strip, by moving the strips in the gluing, we see the result is simply A, B, C, D with 3 strips (or 1-handles) attached. On the other hand, the filling surface $S_{\phi_h(\rho)}$ is $A \leftrightarrow D$ and $B \leftrightarrow C$ glued via a half-twisted strip (FIGURE 2.1.3 (right)), again the picture shows $S_{\phi_h(\rho)}$ is A, B, C, D attached with 3 strips. Hence, we conclude that the 2 filling surfaces S_ρ and $S_{\phi_h(\rho)}$ are homeomorphic relative to the boundary pieces at $x = x_L$ and $x = x_R$. Note also that, this homeomorphism is orientation-preserving if S_ρ is orientable.

When a, b are both the switches of ρ, we define $\phi_h(\rho)$ to be the corresponding m-graded normal ruling having both a, b as switches. A similar argument proves the proposition.

If h is a type I Legendrian Reidemeister move (FIGURE 1.1.2 (left)). Under a type I Legendrian Reidemeister move, the additional crossing necessarily has degree 0. Given a m-graded normal ruling ρ, we define $\phi_h(\rho)$ to be the corresponding normal ruling having this additional crossing as a switch, and vice versa. The proposition holds trivially, since adding one disk along the boundary doesn't change the topological type (and also the orientability) of a surface.

If h is a type II Legendrian Reidemeister move (FIGURE 1.1.2 (middle)). The defining conditions of a normal ruling ensures that the 2 additional crossings can not be switches, so ϕ_h is the obvious bijection. Again, the proposition holds trivially.

If h is a type III Legendrian Reidemeister move (FIGURE 1.1.2 (right)). Let a, b, c be the 3 crossings involved in the Legendian Reidemeister move III. By a smooth isotopy as proved above, we may assume a, b, c are neighboring crossings (see FIGURE 2.1.4 for an example). Given a m-graded normal ruling ρ of T before the move, we need to construction the bijection ϕ_h case by case. If ρ has at most one switch at a, b, c, $\phi_h(\rho)$ is the obvious normal ruling corresponding to ρ, with the same switches at a, b, c as ρ. The proposition follows easily.

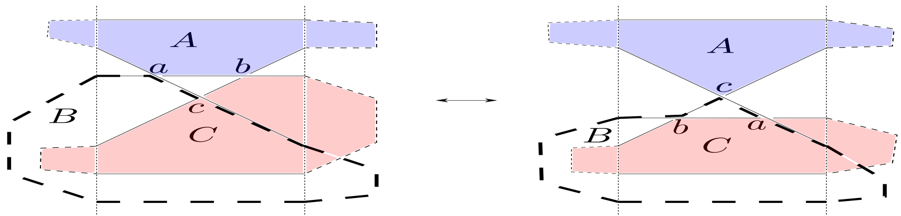


Figure 2.1.4: The bijection between m-graded normal Rulings under a Legendrian Reidemeister move III: In the left figure, the normal ruling ρ has switches at a, b, A, B, C are the 3 eyes containing a, b, c, with A the blue disk, C the red disk and B the disk enclosed by thick dashed lines. In the right figure, the corresponding normal ruling $\phi_h(\rho)$ has switches at b, c, the other letters are similar. By moving the strips at the switches in the vertical strip, the fillings surfaces $S_\rho, S_{\phi_h(\rho)}$ are homeomorphic relative to the boundaries at $x = x_L$ and $x = x_R$.

If ρ has 2 switches at a, b, no switch at c. Then the switches belong to 3 eyes, called A, B, C. We look at the relative positions of the 6 paths of the eyes A, B, C in the small vertical strip containing a, b, c. We only consider one such a case as in FIGURE 2.1.4 (left), the other cases are similar. Note that a, b are switches imply that all the 3 crossings a, b, c have degree 0 modulo m. Moreover, the normal conditions in the definition 2.1.3 of a m-graded normal ruling ensures that, there's a unique way to construct a m-graded normal ruling $\phi_h(\rho)$ which coincides with ρ outside the vertical strip. The converse is also true for the same reason. The picture of ϕ_h is shown in FIGURE 2.1.4 (right), note that now $\phi_h(\rho)$ has 2 switches at b, c, no switch at a. Moreover, the ruling surfaces S_ρ and $S_{\phi_h(\rho)}$ only differ by the gluing of the 3 eyes A, B, C inside the vertical strip. Use the notations as in the proof for smooth isotopies, the left hand side of FIGURE 2.1.4 gives $-B \leftrightarrow A \leftrightarrow -C$, where $-B$ (resp. $-C$) means B with the opposite orientation as that induced from $U \times \mathbb{R}_z$. On the other hand, the right hand side gives $A \leftrightarrow -(B \leftrightarrow C)$. By moving the gluing strips (or 1-handles), it's easy to see that the results after gluing can be identified by an (orientation preserving) homeomorphism which is identity outside the vertical strip. This shows that S_ρ and $S_{\phi_h(\rho)}$ are homeomorphic relative to the boundaries at $x = x_L$ and $x = x_R$, and the homeomorphism is orientation-preserving if S_ρ is orientable.

If the normal ruling ρ has 2 switches at a, c, and no switch at b. Similarly, we look at the relative positions of the 3 eyes A, B, C containing a, b, c. Again, we only look at one such a case as in FIGURE 2.1.5 (left). The other cases are similar. Now by a similar argument as above, FIGURE 2.1.5 proves the proposition. Note, in this case $\phi_h(\rho)$ has 2 switches at a, b, no switch at c. The gluing of the 3 eyes on the left figure is $A \leftrightarrow -(B \leftrightarrow C)$, the gluing of the 3 eyes on the right figure is $-(B \leftrightarrow C) \leftrightarrow A$. They can be identified without changing the parts outside the vertical strip.

The case when ρ has switches at b, c is entirely similar to the case above.

If the normal ruling ρ has switches at all the 3 crossings a, b, c (see FIGURE 2.1.6 for

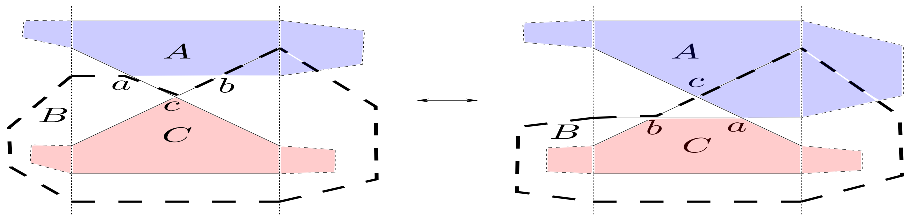


Figure 2.1.5: The bijection between m-graded normal Rulings under a Legendrian Reidemeister move III: In the left figure, the normal ruling ρ has switches at a, c. In the right figure, the corresponding normal ruling $\phi_h(\rho)$ has switches at a, b. the other letters are similar as in FIGURE 2.1.4. By moving the strips at the switches in the vertical strip, the fillings surfaces $S_\rho, S_{\phi_h(\rho)}$ are homeomorphic relative to the boundaries at $x = x_L$ and $x = x_R$.

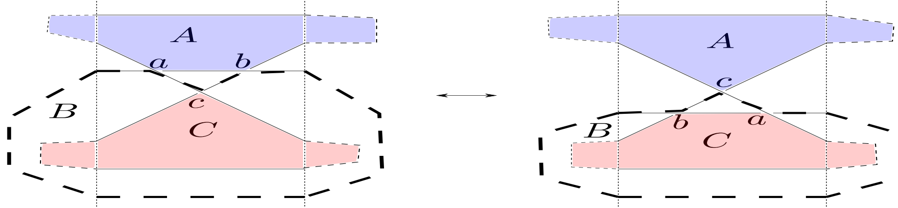


Figure 2.1.6: The bijection between m-graded normal Rulings under a Legendrian Reidemeister move III: the normal ruling ρ (left) and the corresponding normal ruling $\phi_h(\rho)$ (right) both have switches at the 3 crossings a, b, c.

an example), then $\phi_h(\rho)$ is the obvious normal ruling having a, b, c as switches and the same shape as ρ outside the vertical strip. The proposition again follows easily. This finishes the proof.

□

Remark 2.1.12. *Given any m-graded normal ruling ρ of a Legendrian tangle T, in the proof of the previous lemma, a direct check also shows that $s(\phi_h(\rho)) - s(\rho), r(\phi_h(\rho)) - r(\rho)$ and $d(\phi_h(\rho)) - d(\rho)$ (see Definition 2.1.4) are all independent of ρ. We will use this fact in the proof of Corollary 3.2.6.*

Example

Example 2.1.13. *Consider the Legendrian tangle (front) T given by FIGURE 2.1.2 (left), obtained by removing the left and right cusps of a Legendrian trefoil knot. T has 3 crossings a_1, a_2, a_3, and with the $\mathbb{Z}$-valued Maslov potential μ chosen as in the figure, the degrees are $|a_1| = |a_2| = |a_3| = 0$. Moreover, T has 4 left end-points, 4 right end-points and no left or right cusps. So in Remark 2.1.8, $n_L = n_R = 4, c_L = c_R = 0$. The left piece T_L (resp. the* *right piece* T_R*) consists of 4 parallel lines, labeled from top to bottom, say, by* $1, 2, 3, 4$*, with Maslov potential values* $2, 1, 1, 0$ *respectively.*

Let's calculate the Ruling polynomials for (T, μ)*. Firstly, let* $m \neq 1$ *be a nonnegative integer, then the set of* m*-graded normal rulings for* T_L *(resp.* T_R*) is* $NR^m_{T_L} = \{(\rho_L)_1 = (12)(34), (\rho_L)_2 = (13)(24)\}$ *(resp.* $NR^m_{T_R} = \{(\rho_R)_1 = (12)(34), (\rho_R)_2 = (13)(24)\}$*). Here, for example in* T_L $(12)(34)$ *means the pairing between the strands* $1, 2$ *(resp.* $3, 4$*), corresponding to a* m*-graded normal ruling of* T_L*.*

Using the notations before Lemma 2.1.9 and in Remark 2.1.5, the m*-graded normal rulings of* (T, μ) *are:*

(1). $NR^m_T((\rho_L)_1, (\rho_R)_1) = \{\rho_3 = \{a_1\}; \rho_5 = \{a_3\}; \rho_8 = \{a_1, a_2, a_3\}\}$*;*

(2). $NR^m_T((\rho_L)_1, (\rho_R)_2) = \{\rho_1 = \emptyset; \rho_6 = \{a_1, a_2\}\}$*;*

(3). $NR^m_T((\rho_L)_2, (\rho_R)_1) = \{\rho_2 = \emptyset; \rho_7 = \{a_2, a_3\}\}$*;*

(4). $NR^m_T((\rho_L)_2, (\rho_R)_2) = \{\rho_4 = \{a_2\}\}$*.*

Note that $\{a_1, a_3\}$ *is not a* m*-graded normal ruling since it violates the normal condition in the definition 2.1.3. Apply the computation formula in Remark 2.1.8, the Ruling polynomials of* (T, μ) *and the corresponding maximal degrees* d *in* z *are:*

(1). $<(\rho_L)_1|R^m_T(z)|(\rho_R)_1>= 2z + z^3$*, and* $d = 3$*;*

(2). $<(\rho_L)_1|R^m_T(z)|(\rho_R)_2>= 1 + z^2$*, and* $d = 2$*;*

(3). $<(\rho_L)_2|R^m_T(z)|(\rho_R)_1>= 1 + z^2$ *with* $d = 2$*;*

(4). $<(\rho_L)_2|R^m_T(z)|(\rho_R)_2>= z$ *with* $d = 1$*.*

Note that for $m = 1$*, each of the 2 sets* $NR^1_{T_L}$ *and* $NR^1_{T_R}$ *contains an additional* 1*-graded normal ruling* $(\rho_L)_3$ *(resp.* $(\rho_R)_3$*)* $= \{(14)(23)\}$*. However, no* 1*-graded normal ruling of* T *restricts to* $(\rho_L)_3$ *or* $(\rho_R)_3$*, hence the above formula also computes the* 1*-graded ruling polynomials for* (T, μ)*.*

In Example 3.2.13, as an illustration of the main theorem 3.2.7, we will see that the ruling polynomials with boundary conditions calculated here, indeed match with the augmentation numbers with the same boundary conditions (to be defined in Section 3.1) for the Legendrian tangle (T, μ) *as above.*

2.2 The LCH differential graded algebras for Legendrian tangles

Generalizing the Chekanov-Eliashberg construction of the LCH DGAs for Legendrian links, the LCH DGAs for Legendrian tangles with simple fronts (See Section 1.1) were explicitly constructed in [32]. Here we give the basic constructions and properties of the LCH DGAs associated to any Legendrian tangle T (not necessarily with simple front). The key properties of the DGAs are the homotopy invariance and co-sheaf property. As in the Section 1.2, we allow some base points placed on the tangle.

Given an oriented tangle front T, provided with a $\mathbb{Z}/2r$-valued Maslov potential μ. We can orient the tangle so that each strand is right-moving (resp. left-moving) if and only if its Maslov potential value is even (resp. odd). We place some base points $*_1, \ldots, *_B$ on T so that each connected component containing a right cusp has at least one base point. Assume T has n_L left end-points and n_R right end-points, labeled from top to bottom by $1, 2, \ldots, n_L$ (resp. $1, 2, \ldots, n_R$). We will construct a LCH DGA associated to the resolution (see Section 1.1) of T. The idea is to embed the tangle front T into a Legendrian link front Λ, and take the resolution of Λ. Then define the $\mathbb{Z}/2r$-graded LCH DGA $\mathcal{A}(T) = \mathcal{A}(T, \mu, *_1, \ldots, *_B)$ as a sub-DGA of $\mathcal{A}(\mathrm{Res}(\Lambda))$.

Embed a Legendrian tangle into a Legendrian link

Let $(T, \mu, *_1, \ldots, *_B)$ be given as above. In this subsection, we will give the construction to embed T into a Legendrian link front (see FIGURE 2.2.1 for an illustration). In the case of FIGURE 2.2.1 (left), T is the Legendrian tangle in the vertical strip, with $n_L = 4, n_R = 2$ and $B = 2$.

We firstly glue a n_L-copy of a left cusp along the top end-points to the n_L left end-points of T (see the 4-copy of the left cusp with crossings α_{ij}'s in FIGURE 2.2.1 (left)), and also glue a n_R-copy of a right cusp along the bottom end-points to the n_R right end-points of T (See the 2-copy of the right cusp with crossing β_1 in FIGURE 2.2.1 (left)). Next, we glue a n_R-copy of a left cusp, placed to the left of the diagram, along the top end-points to the top end-points of the n_R-copy of the right cusp (See the 2-copy left cusp to the left of the diagram in FIGURE 2.2.1 (left)). Now we are left with a diagram, say D, with $n_L + n_R$ right end-points (see the bottom dashed line in FIGURE 2.2.1 (left)). We glue these right end-points via $n_L + n_R$ right cusps as follows. We extend the Maslov potential μ to D, this extension is unique. Every connected component with nonempty boundary (i.e. component which is not a loop) of T is connected to exactly 2 such right end-points, and it's easy to see that $\mu(\text{upper end-point}) - \mu(\text{lower end-point}) = \pm 1 (\text{mod} 2r)$. We glue a right cusp to the 2 end-points from the right so that μ defines a $\mathbb{Z}/2r$-valued Maslov potential on the resulting front diagram. We will place these right cusps so that they have almost the same x-coordinates. Note that this procedure may involve some additional crossings (See the bottom-right in FIGURE 2.2.1 (left)).

Let's denote by Λ the resulting front diagram, we will also add some additional base points, for example one base point at each of the additional $n_R + n_L$ right cusps in the bottom of Λ, so that each component of Λ contains at least one base point. By construction, Λ is equipped with a $\mathbb{Z}/2r$-valued Maslov potential, still denoted by μ. Moreover, Λ is simple (see Section 1.1) away from T. The n_L-copy of the left cusp glued to the left end-points of T

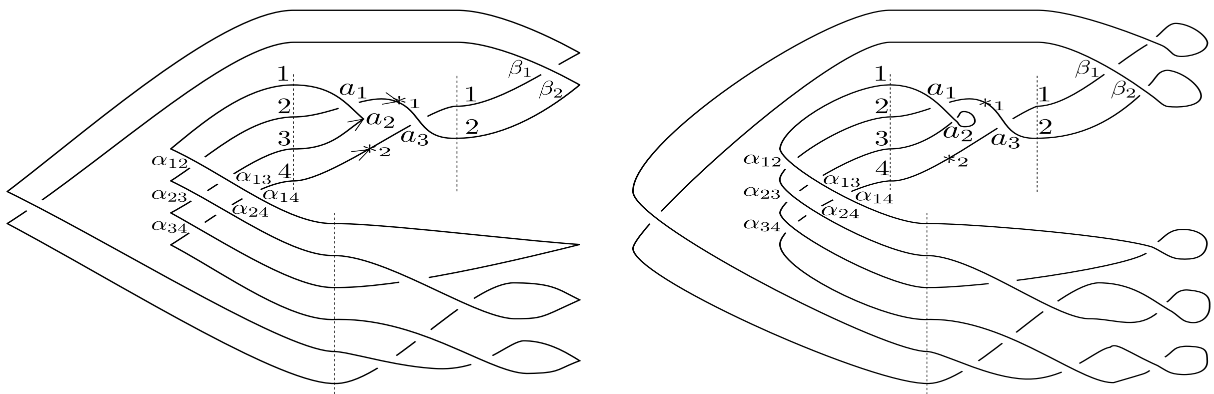


Figure 2.2.1: LCH DGA for tangles: In the left picture, embed the Legendrian tangle T (the part in the vertical strip) into a Legendrian link front Λ; In the right picture, take the resolution of the Legendrian link front Λ.

has $\binom{n_L}{2}$ crossings, denoted by $\alpha_{ij}, 1 \leq i < j \leq n_L$, where α_{ij} is the crossing of the 2 strands connected to the 2 left end-points i, j of T. Then, we have $|\alpha_{ij}| = \mu(i) - \mu(j) - 1$.

Now by the resolution construction, we can define the LCH DGA $\mathcal{A}(\mathrm{Res}(\Lambda))$. Let $\{a_1, a_2, \ldots, a_R\}$ be the crossings and right cusps of T, $t_1, t_2, \ldots, t_B$ be the generators in $\mathcal{A}(\mathrm{Res}(\Lambda))$ corresponding to the base points $*_1, *_2, \ldots, *_B$. By the resolution construction [24], the differential ∂a_i only involves the generators $\{a_j, 1 \leq j \leq R, t_j^{\pm 1}, 1 \leq j \leq B\}$ and $\{\alpha_{ij}, 1 \leq i < j \leq n_L\}$.

Moreover, the differentials of α_{ij}'s are given by

$$\partial \alpha_{ij} = \sum_{i<k<j} (-1)^{|\alpha_{ik}|+1} \alpha_{ik} \alpha_{kj}.$$

As a consequence, the subalgebra generated by $t_1^{\pm 1}, \ldots, t_B^{\pm 1}, a_1, \ldots, a_R$ and $\alpha_{ij}, 1 \leq i < j \leq n_L$ form a sub-DGA of $\mathcal{A}(\mathrm{Res}(\Lambda))$. This leads to the definition of the LCH DGA $\mathcal{A}(T)$ of the Legendrian tangle front T.

LCH DGAs via Legendrian tangle fronts

Now, let's translate the construction of sub-DGAs in the previous subsection into definitions involving only T.

The general definition

Definition/Proposition 2.2.1. *Define the $\mathbb{Z}/2r$-graded LCH DGA $\mathcal{A}(T)$ as follows:*

As an algebra: $\mathcal{A}(T) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_B^{\pm 1}] < a_i, 1 \leq i \leq R, a_{ij}, 1 \leq i < j \leq n_L >$ *is a free associative algebra over* $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_B^{\pm 1}]$*, where* a_{ij} *corresponds to the pair of left end-points* i, j *of* T.

The grading and differential *is induced from the identification of* $\mathcal{A}(T)$ *with the sub-DGA obtained above, via* $t_i^{\pm 1} \leftrightarrow t_i^{\pm 1}, a_i \leftrightarrow a_i$ *and* $a_{ij} \leftrightarrow \alpha_{ij}$. *By the construction above, the DGA* $\mathcal{A}(T)$ *is* independent of the choices *involved in the construction of* Λ. *In particular, we can translate the DGA* $\mathcal{A}(T)$ *purely in terms of the combinatorics of the tangle front* T. *More precisely, we have:*

The grading: $|t_i^{\pm 1}| = 0, |a_i| = \mu(\textit{over-strand}) - \mu(\textit{under-strand})$ *if* a_i *is a crossing,* $|a_i| = 1$ *if* a_i *is a right cusp, and* $|a_{ij}| = \mu(i) - \mu(j) - 1$.

The differential: *As usual, we impose the graded Leibniz Rule* $\partial(x \cdot y) = (\partial x) \cdot y + (-1)^{|x|} x \cdot \partial y$ *and the differential of the generators are defined as follows:* $\partial(t_i^{\pm 1}) = 0$; *The differential of* a_{ij} *given by the same formula for* α_{ij} *with* $\alpha_{\bullet,\bullet}$ *replaced by* $a_{\bullet,\bullet}$, *that is,*

$$\partial a_{ij} = \sum_{i<k<j} (-1)^{|a_{ik}|+1} a_{ik} a_{kj}. \tag{2.2.0.1}$$

To translate the differential of a crossing or a right cusp, we proceed as in [24, Def.2.6]. Let $a = a_i$ *and* $v_1, \ldots, v_n$ *be some elements in the generators* $\{a_i, 1 \leq i \leq R, a_{ij}, 1 \leq i < j \leq n_L\}$ *of* T *for* $n \geq 0$. *Let* $D_n^2 = D^2 - \{p, q_1, \ldots, q_n\}$ *be a fixed oriented disk with* $n+1$ *boundary punctures (or vertices)* $p, q_1, \ldots, q_n$*, arranged in a counterclockwise order.*

Definition 2.2.2. *Define the moduli space* $\Delta(a; v_1, \ldots, v_n)$ *to be the space of* admissible disks u *of the tangle front* T *up to re-parametrization, that is,*

(i) (Immersion with singularities) *The map* $u : (D_n^2, \partial D_n^2) \to (\mathbb{R}_{xz}^2, T)$ *is an immersion, orientation-preserving, and smooth away from possible singularities at left and right cusps, near which the image of the map are indicated as in FIGURE 2.2.2.a,b. Note that the singularities are not vertices of* D_n^2*;*

(ii) (Initial/positive vertex) u *extends continuously to* p*, with* $u(p) = a$*, near which the image of the map is indicated as in Figure 2.2.2.c;*

(iii) (Negative vertices at a crossing) *If* v_i *is a crossing,* u *extends continuously to* q_i*, with* $u(q_i) = v_i$*, near which the image of the map is indicated as in Figure 2.2.2.d;*

(iv) (Negative vertices at a right cusp) *If* v_i *is a right cusp,* u *extends continuously to* q_i*, with* $u(q_i) = v_i$*, near which the image of the map is indicated as in Figure 2.2.2.e;*

(v) (Negative vertices at a pair of left end-points) *If* v_i *is a pair of left end-points* a_{jk}*, we require that, as one approaches* q_i *in* D_n^2*,* u *limits to the line segment* $[j, k]$ *at the left boundary between the left end-points* j, k *of* T*;*

(vi) The x*-coordinate on the image* $\overline{u(D_n^2)}$ *has a unique local maximum at* a*.*

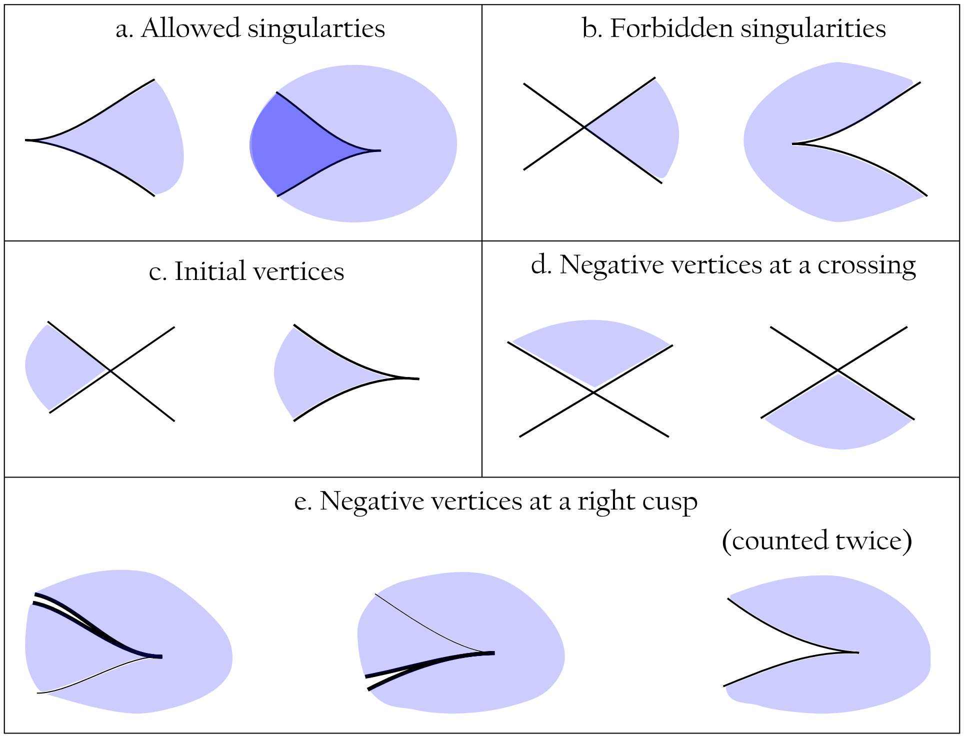


Figure 2.2.2: Admissible disks: The image of the disk D^2_n under an admissible map near a singularity or a vertex on the boundary ∂D^2_n. The first row indicates the possible singularities, the second and third rows indicate the possible vertices. In the first 2 pictures of part e, 2 copies of the same strand (the heavy lines) are drawn for clarity.

Note: the last condition (vi) is in fact a consequence of the previous ones (i)-(v). All the defining conditions are direct translations from those in Definition 1.2.1 via Lagrangian projection, for $\mathcal{A}(\mathrm{Res}(\Lambda))$ in Section 2.2. Via the resolution construction (Figure 1.1.1), the only nontrivial part is the translation for a right cusp, near which the defining conditions are illustrated by Figure 2.2.3.

For each $u \in \Delta(a; v_1, \dots, v_n)$, walk along $\overline{u(\partial D^2_n)}$ starting from a in counterclockwise direction, we encounter a sequence $s_1, \dots, s_N (N \geq n)$ of negative vertices of u (crossings, right cusps, or pairs of left end-points as in Definition 2.2.2) and base points (away from the previous negative vertices). Translate Definition 1.2.2, we obtain

Definition 2.2.3. *The* weight *of u is $w(u) := w(s_1) \dots w(s_N)$, where*

(i) $w(s_k) = t_i$(resp. t_i^{-1}) if s_k is the base point $*_i$, and the boundary orientation of $u(\partial D^2)$ agrees (resp. disagrees) with the orientation of T near $*_i$. Note that this includes the

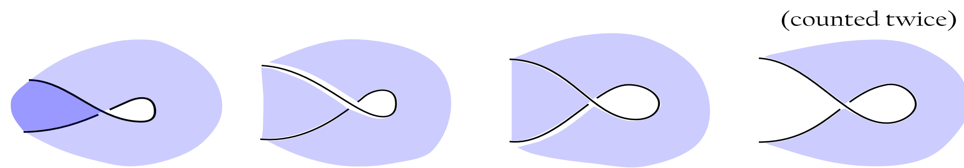


Figure 2.2.3: The singularity and negative vertices at a right cusp after resolution: The first figure corresponds to a singularity (Figure 2.2.2.a), the remaining ones correspond to a negative vertex (FIGURE 2.2.2.e, going from left to right).

case when the base point $*_i$ is located at a right cusp, which is also a singularity of u (See Figure 2.2.2.a);

(ii) $w(s_k) = v_i$ *(resp.* $(-1)^{|v_i|+1}v_i$*) if* s_k *is the crossing* v_i *and the disk* $\overline{u(D_n^2)}$ *occupies the top (resp. bottom) quadrant of* v_i *(See Figure 2.2.2.d);*

(iii) $w(s_k) = a_{ij}$ *if* s_k *is the pair of left end-points* a_{ij}*;*

(iv) $w(s_k) = w_1(s_k)w_2(s_k)$ *if* s_k *is the right cusp* $v_i = u(q_i)$ *(see Figure 2.2.2.e), where* $w_2(s_k) = v_i$ *(resp.* v_i^2*) if the image of* u *near* q_i *looks like the first two diagrams (resp. the third diagram) of Figure 2.2.2.e;*
$w_1(s_k) = 1$ *if* s_k *is a unmarked right cusp (equipped with no base point);*
$w_1(s_k) = t_j$ *(resp.* t_j^{-1}*) if* v_i *is a marked right cusp equipped with the base point* $*_j$*, and* v_i *is an up (resp. down) right cusp*[2]*. See Figure 2.2.3 for an illustration.*

Notice that the convention for the orientation signs *here is as follows: At each crossing of even degree of the tangle front* T*, the two quadrants to the lower right of the under-strand have negative orientation signs. All other quadrants have positive orientation signs.*

Definition 2.2.4. *For* $a = a_i$ *a crossing or a right cusp, its differential is given by*

$$\partial a = \sum_{n, v_1, \ldots, v_n} \sum_{u \in \Delta(a; v_1, \ldots, v_n)} w(u) \qquad (2.2.0.2)$$

where for $a = a_i$ *a right cusp, we also include the contribution from an "invisible" disk* u *coming from the resolution construction (see Section 1.1), with* $w(u) = 1$ *(resp.* t_j^{-1} *or* t_j*), if there's no base point (resp. a base point* $*_j$*, depending on whether* a_i *is an up or down right cusp).*

Example 2.2.5. *Let* T *be the Legendrian tangle in Figure 2.2.1 (left), with a choice of* $\mathbb{Z}/2r$*-valued Maslov potential* μ*. As an algebra, the LCH DGA is* $\mathcal{A}(T) = \mathcal{A}(T, \mu, *_1, *_2) =$

[2]Recall that a cusp is called *up* (resp. *down*) if the orientation of the front T near the cusp goes up (resp. down).

$\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}] < a_1, a_2, a_3, a_{ij}, 1 \leq i < j \leq 4 >$, where as usual the a_{ij}'s are the generators corresponding to the pairs of left endpoints. The differential ∂ for $\mathcal{A}(T)$ is: As usual, $\partial a_{ij} = \sum_{i<k<j}(-1)^{|a_{ik}|+1}a_{ik}a_{kj}$ and $\partial(t_i^{\pm1}) = 0$. The differentials for a_i's are as follows

$$\partial a_1 = a_{12};$$
$$\partial a_2 = 1 + a_{13} + (-1)^{|a_1|+1}a_1a_{23};$$
$$\partial a_3 = t_1^{-1}(a_{24} + a_{23}(a_{14} + (-1)^{|a_1|+1}a_1a_{24} + a_2a_{34}))t_2.$$

Note: there's a strategy to compute ∂a_3. We can cut T into elementary Legendrian tangles and apply Definition/Proposition 2.2.9.

By embedding the Legendrian tangle T into a Legendrian link, the proof of Theorem 1.2.5 also shows that

Proposition 2.2.6. *The isomorphism class of $\mathcal{A}(T)$ is independent of the locations of the base points on each connected component of T. The stable isomorphism class of $\mathcal{A}(T)$ is invariant under Legendrian isotopy of T.*

LCH DGA for simple Legendrian tangles

In the case when T is a simple Legendrian tangle (see Section 1.1 for the definition), in particular when T is nearly plat, we have a simple description of $\mathcal{A}(T)$.

The algebra and grading are the same as before, but the differential counts simpler objects. More precisely, for $a = a_i$ a crossing or a right cusp, the differential ∂a is given by the same formula as in Definition 2.2.4. However, we have

Lemma 2.2.7 ([24, §.2.3])**.** *For T a simple Legendrian tangle front, any admissible disk u in $\Delta(a; v_1, \ldots, v_n)$ must satisfy:*

1. *u is an* embedding*, not just an immersion, so no singularity at a right cusp (see Figure 2.2.2.a);*
2. *Each negative vertex of u must be a crossing, so there's no negative vertex at a right cusp (see Figure 2.2.2.e);*
3. *There's at most one negative vertex at a pair of left end-points.*

Example 2.2.8. *Consider the Legendrian tangle (T, μ) in Example 2.1.13 (See Figure 2.1.2 (left)). As usual, label the left end points of T by $1, 2, 3, 4$ from top to bottom. Let $a_{ij}, 1 \leq i < j \leq 4$ be the pairs of left end points of T. Then the LCH DGA $\mathcal{A}(T)$ is generated by a_{ij} and a_1, a_2, a_3, with the grading: $|a_{12}| = |a_{13}| = |a_{24}| = |a_{34}| = 0, |a_{23}| = -1, |a_{14}| = 1$ and $|a_1| = |a_2| = |a_3| = 0$. The differential is given by*

$$\partial a_{ij} = \sum_{i<l<j}(-1)^{|a_{il}|+1}a_{il}a_{lj};$$
$$\partial a_1 = a_{23};$$
$$\partial a_2 = \partial a_3 = 0.$$

The co-sheaf property

Let T be a Legendrian tangle in J^1U. Let V be an open subinterval of U such that, the boundary $(\partial\overline{U}) \times \mathbb{R}_z$ is disjoint from the crossings, cusps and base points of T. $T|_V$ then gives a Legendrian tangle in J^1V with Maslov potential induced from that of T, hence the LCH DGA $\mathcal{A}(T|_V)$ is defined. There's indeed a co-restriction map of DGAs.

Definition/Proposition 2.2.9 ([26, Prop.6.12],[32])**.** *The following defines a morphism of $\mathbb{Z}/2r$-graded DGAs $\iota_{UV} : \mathcal{A}(T|_V) \to \mathcal{A}(T)$:*

(1) ι_{UV} sends a generator of $\mathcal{A}(T|_V)$, corresponding to a crossing, a right cusp or a base point of T, to the corresponding generator of $\mathcal{A}(T)$;

(2) For a generator b_{ij} in $\mathrm{A}(T|_V)$ corresponding to the pair of left end-points i, j of $T|_V$, the image $\iota_{UV}(b_{ij})$ is defined as follows:
Use the notations in Section 2.2.2 and consider the moduli space $\Delta(b_{ij}; v_1, \ldots, v_n)$ of disks $u : (D^2_n, \partial D^2_n) \to (\mathbb{R}^2_{xz}, T)$ satisfying the conditions in definition 2.2.2, with the condition for a there replaced by "u limits to the line segments $[i, j]$ between the pair of left end-points i, j of $T|_V$ at the puncture $p \in \partial D^2$ and u attains its local maxima exactly along $[i, j]$". Then define

$$\iota_{UV}(b_{ij}) = \sum_{n, v_1, \ldots, v_n} \sum_{u \in \Delta(b_{ij}, v_1, \ldots, v_n)} w(u) \tag{2.2.0.3}$$

Proof. Apply the proof of Prop. 6.12 in [26]: Though it only deals with Legendrian tangles in nearly plat positions, essentially the same arguments work in the general case, with 'embedded disks' replaced by 'immersed disks' everywhere. □

Remark 2.2.10. *From the definition, it's easy to see that if the left boundary of V coincides with that of U, then the co-restriction map $\iota_{UV} : \mathcal{A}(T|V) \hookrightarrow \mathcal{A}(T)$ is an inclusion.*

Example 2.2.11 (co-restriction ι_R for a right cusp)**.** *One key example for the co-restriction of DGAs is $\iota_R : \mathcal{A}(T_R) \to \mathcal{A}(T)$, where T be an elementary Legendrian tangle of a single (marked or unmarked) right cusp a, and T_R is the right piece of T. For simplicity, assume T has 4 left endpoints and 2 right endpoints as in Figure 2.2.4. Then $\mathcal{A}(T_R) = \mathbb{Z} < b_{12} >$, where b_{12} is the generator corresponding to the pair of left endpoints of T_R, and $\mathcal{A}(T) = \mathbb{Z}[t, t^{-1}] < a, a_{ij}, 1 \leq i < j \leq 4 >$ with $\partial a = t^{\sigma(a)} + a_{23}$ (see Definition 2.2.12 below), where a_{ij}'s correspond to the pairs of left endpoints of T, t is the generator corresponding to the base point if the right cusp is marked and $t = 1$ otherwise. Then $\iota_R : \mathcal{A}(T_R) \to \mathcal{A}(T)$ is given by*

$$\begin{aligned} \iota_R(b_{12}) &= a_{14} + a_{13}t^{-\sigma(a)}a_{24} + a_{12}at^{-\sigma(a)}a_{24} + a_{13}t^{-\sigma(a)}aa_{34} + a_{12}at^{-\sigma(a)}aa_{34} \\ &= a_{14} + t^{-\sigma(a)}(a_{13} + a_{12}a)(a_{24} + aa_{34}). \end{aligned}$$

We introduce a sign at a right cusp, which will also be used later (see Lemma 3.3.2).

Figure 2.2.4: Left: An elementary Legendrian tangle of an unmarked right cusp. Right: An elementary Legendrian tangle of a marked right cusp.

Definition 2.2.12. *Given a right cusp a of the oriented tangle front T, we define the* sign $\sigma = \sigma(a)$ *of a to be* 1 *(resp. -1) if a is a down (resp. up) cusp. See Figure 2.2.5.*

Figure 2.2.5: Left: a down right cusp. Right: an up right cusp.

One key property of LCH DGAs for Legendrian tangles is the co-sheaf property:

Proposition 2.2.13 ([26, Thm.6.13])**.** *If $U = L \cup_V R$ is the union of 2 open intervals L, R with non-empty intersection V, then the diagram of co-restriction maps*

$$\begin{array}{ccc} \mathcal{A}(T|_V) & \xrightarrow{\iota_{RV}} & \mathcal{A}(T|_R) \\ {\scriptstyle \iota_{LV}}\downarrow & & \downarrow{\scriptstyle \iota_{UR}} \\ \mathcal{A}(T|_L) & \xrightarrow{\iota_{UL}} & \mathcal{A}(T) \end{array} \tag{2.2.0.4}$$

gives a pushout square of $\mathbb{Z}/2r$-graded DGAs.

Proof. Again the same argument in the proof of Theorem 6.13 in [26] (The case for Legendrian tangles in nearly plat positions) applies to the general case. □

Chapter 3

Points-counting of augmentation varieties

In this chapter, we study the representation varieties (of rank 1) (called augmentation varieties) of the LCH DGAs associated to Legendrian tangles. We firstly define augmentation varieties and augmentation numbers (with boundary conditions) associated to Legendrian tangles, generalizing those of Legendrian knots. We then show a ruling decomposition for the augmentation varieties, and show that their points-counting over finite fields (or augmentation numbers) are computed by ruling polynomials defined in chapter 2. These generalize and provide more natural proofs to the previous results of M.Henry and D.Rutherford [15].

3.1 Augmentation varieties for Legendrian tangles

Fix a Legendrian tangle T, with $\mathbb{Z}/2r$-valued Maslov potential μ, base points $*_1,\ldots,*_B$ so that each connected component containing a right cusp has at least one base point. Denote the crossings, right cusps and pairs of left end-points by $\mathcal{R}=\{a_1,\ldots,a_N\}$. As always, the base points are assumed to be away from the crossings and left cusps of T. Let n_L, n_R be the numbers of left and right end-points in T respectively.

We define the LCH DGA $(\mathcal{A}(T),\partial)$ as in the previous Section. So as an associative algebra we have $\mathcal{A}:=\mathcal{A}(T)=\mathbb{Z}[t_1^{\pm 1},\ldots,t_B^{\pm 1}]<a_1,\ldots,a_N>$. Fix a nonnegative integer m dividing r and a base field k.

Definition 3.1.1. *A* m-graded (or $\mathbb{Z}/m$-graded) k-augmentation *of* $\mathcal{A}$ *is an unital algebraic map* $\epsilon:(\mathcal{A},\partial)\rightarrow(k,0)$ *such that* $\epsilon\circ\partial=0$*, and for all* a *in* $\mathcal{A}$ *we have* $\epsilon(a)=0$ *if* $|a|\neq 0(\mathrm{mod}\, m)$*. Here* $(k,0)$ *is viewed as a DGA concentrated on degree 0 with zero differential. Morally, "*ϵ *is a* $\mathbb{Z}/m\mathbb{Z}$*-graded DGA map".*

Definition 3.1.2. *Define* $\mathrm{Aug}_m(T,k)$ *to be the set of* m*-graded* k*-augmentations of* $\mathcal{A}(T)$*. This defines an affine subvariety of* $(k^{\times})^B\times k^N$*, via the map*

$$\mathrm{Aug}_m(T,k)\ni\epsilon\rightarrow(\epsilon(t_1,\ldots,\epsilon(t_B),\epsilon(a_1),\ldots,\epsilon(a_N)))\in(k^{\times})^B\times k^N$$

with the defining polynomial equations $\epsilon \circ \partial(a_i) = 0, 1 \leq i \leq N$ *and* $\epsilon(a_i) = 0$ *for* $|a_i| \neq 0(\text{mod}m)$. *This affine variety* $\text{Aug}_m(T, k)$ *will be called the (full)* m-graded augmentation variety *of* $(T, \mu, *_1, \ldots, *_B)$.

Example 3.1.3 (The augmentation variety for trivial Legendrian tangles)**.** *Let* T *be the trivial Legendrian tangle of* n *parallel strands, labeled from top to bottom by* $1, 2, \ldots, n$, *equipped a* $\mathbb{Z}/2r$*-valued Maslov potential* μ. *The LCH DGA is* $\mathcal{A}(T) = \mathbb{Z} < a_{ij}, 1 \leq i < j \leq n >$, *with the grading* $|a_{ij}| = \mu(i) - \mu(j) - 1$ *and the differential given by formula (2.2.0.1). The* m*-graded augmentation variety* $\text{Aug}_m(T; k)$ *is*

$$\text{Aug}_m(T; k) = \{(\epsilon(a_{ij}))_{1 \leq i < j \leq n} | \epsilon \circ \partial a_{ij} = 0, \textit{ and } \epsilon(a_{ij}) = 0 \textit{ if } |a_{ij}| \neq 0(\text{mod}m).\}$$

On the other hand,

Definition 3.1.4. *Associate to the trivial Legendrian tangle* (T, μ), define *a canonical* $\mathbb{Z}/m$*-graded filtered* k*-module* $C = C(T)$*:* C *is the free* k*-module generated by* $e_1, \ldots, e_n$ *corresponding to the* n *strands of* T *with grading* $|e_i| = \mu(i)(\text{mod}m)$. *Moreover,* C *is equipped with a decreasing filtration* $F^0 \supset F^1 \supset \ldots \supset F^n : F^iC = \text{Span}\{e_{i+1}, \ldots, e_n\}$.
Define $B_m(T) := \text{Aut}(C)$ *to be the automorphism group of the* $\mathbb{Z}/m$*-graded filtered* k*-module* C. Denote $I = I(T) := \{1, 2, \ldots, n\}$.

Now, in the example, given any m*-graded augmentation* ϵ *for* $\mathcal{A}(T)$, *we construct a* $\mathbb{Z}/m$*-graded chain complex* $C(\epsilon) = (C, d(\epsilon))$*: The differential* $d = d(\epsilon)$ *is filtration preserving, of degree* -1 *given by*

$$< de_i, e_j >= 0 \textit{ for } i \geq j \textit{ and } < de_i, e_j >= (-1)^{\mu(i)}\epsilon(a_{ij}) \textit{ for } i < j.$$

Here $< de_i, e_j >$ *denotes the coefficient of* e_j *in* de_i. *The condition that* d *is of degree* -1 *is equivalent to:* $< de_i, e_j >= (-1)^{\mu(i)}\epsilon(a_{ij}) = 0$ *if* $\mu(i) - \mu(j) - 1 = |a_{ij}| \neq 0(\text{mod}m)$ *for all* $i < j$. *The condition of the differential* $d^2 = 0$ *is equivalent to: for all* $i < j$ *have* $< d^2e_i, e_j >= \sum_{i<k<j} < de_i, e_k >< de_k, e_j >= 0$, *i.e.* $\sum_{i<k<j}(-1)^{\mu(i)-\mu(k)}\epsilon(a_{ik})\epsilon(a_{kj}) = \epsilon \circ \partial a_{ij} = 0$.

Thus, we see that the map $\epsilon \to C(\epsilon)$ *gives an* isomorphism *between the augmentation variety* $\text{Aug}_m(T; k)$ *and the set* $MCS_m^A(T; k)$ *of* $\mathbb{Z}/m$*-graded filtered chain complexes* (C, d), *or equivalently, the set of filtration preserving degree* -1 *differentials* d *of* C. *From now on, we will always use this identification (see also Section 3.3).*

Given the Legendrian tangle (T, μ) of n parallel strands, $B_m(T)$ acts on $\text{Aug}_m(T; k) = MCS_m^A(T)$ via conjugation: given $\varphi \in B_m(T)$ and (C, d) in $MCS_m^A(T; k)$, have $\varphi \cdot (C, d) := (C, \varphi \circ d \circ \varphi^{-1})$. In particular, the $B_m(T)$-orbit $B_m(T) \cdot (C, d)$ (or $B_m(T) \cdot d$) is simply the isomorphism classes of d.

Lemma 3.1.5 (Barannikov normal form, See also [30, 20])**.** *Let* (C, d) *be any* $\mathbb{Z}/m$*-graded filtered chain complex over* k, *where* $C = \text{Span}_k\{e_1, \ldots, e_n\}$ *is fixed with the decreasing* *filtration* $F^0 \supset F^1 \supset \ldots \supset F^n$*:* $F^iC = \mathrm{Span}_k\{e_{i+1}, \ldots, e_n\}$*, then the isomorphism class of* (C, d) *has a unique representative, say* (C, d_0)*, such that the matrix* $(< d_0e_i, e_j >)_{i,j}$ *has at most one nonzero entry in each row and column and moreover these are all* 1*'s. Equivalently, there're* $2k$ *distinct indices* $i_1 < j_1, \ldots, i_k < j_k$ *in* $\{1, \ldots, n\}$ *for some* k*, such that* $d_0e_{i_l} = e_{j_l}$ *for* $1 \leq l \leq k$ *and* $d_0e_i = 0$ *otherwise.*

The unique representative (C, d_0) *is called the* Barannikov normal form *of* (C, d)*.*

Proof. We divide the index set $I := \{1, \ldots, n\}$ into 3 types: *upper*, *lower* and *homological*.

For each $1 \leq i \leq n$, an element of the form $c_ie_i + \sum_{k>i} c_ke_k$ is called *i-admissible* if $c_i \neq 0$ and $c_k = 0$ if $|e_k| \neq |e_i| (\mathrm{mod} m)$ for all $k > i$. In other words, the set of i-admissible elements is the same as $\mathrm{Aut}(C) \cdot e_i$, the image of e_i under the automorphism group of the $\mathbb{Z}/m$-graded filtered k-module C. In particular, any automorphism of C preserves the set of i-admissible elements.

- i is called *d-closed* (or *closed*) if there's an i-admissible element x such that $dx = 0$. Otherwise, i is called *d-upper* (or *upper*).

To check the definition only depends on the isomorphism class of d: If d' is another representative in the isomorphism class of d, so $d' = \varphi \cdot d = \varphi \circ d \circ \varphi^{-1}$ for some $\varphi \in \mathrm{Aut}(C)$. If i is d-closed, say $dx = 0$ with x i-admissible, then $d'\varphi(x) = \varphi(dx) = 0$ with $\varphi(x)$ i-admissible, hence i is also d'-closed.

For each index i, and any i-admissible element x, we can write $dx = *_le_l + \sum_{k>l} *_ke_k$ for some $l > i$ with $*_l \neq 0$, i.e. dx is l-admissible. If $dx = 0$ (that is, i is closed), then $l := \infty$ and "dx is ∞-admissible" means $dx = 0$. Now, *define* $\rho_d(x) := l$.

If $d' = \varphi \cdot d$ is another representative, then $d'\varphi(x) = \varphi(dx)$ is also l-admissible, hence $\rho_{d'}(\varphi(x)) = \rho_d(x)$. For each index i, *define*

$$\rho_d(i) := \max\{\rho_d(x) | x \text{ is } i\text{-admissible}\}$$

By definition, $\rho_d(i) > i$. And, the previous identity shows that $\rho_d(i)$ only depends on the isomorphism class of d. So we can write $\rho(i) = \rho_d(x)$. Also, by definition, i is upper if and only if $\rho(i) < \infty$.

- j is called *lower*, if $j = \rho(i)$ for some upper index i.

If $j = \rho(i)$ is lower, then $j = \rho(x)$ for some i-admissible element x, hence $dx = *_je_j + \sum_{k>j} *_ke_k$ is j-admissible. It follows that $d(*_je_j + \sum_{k>j} *_ke_k) = d^2x = 0$. Therefore, j is *closed*.

- j is called *homological*, if j is closed but not lower.

As a consequence, we obtain a partition and a map associated to the isomorphism class of d

$$I = L \sqcup H \sqcup U \tag{3.1.0.1}$$
$$\rho : U \to L$$

where L, H and U are the sets of lower, homological and upper indices respectively. We emphasize that the partition and the map depend only on the isomorphism class of d. Moreover, $\rho : U \to L$ is a *bijection*. By definition, it's clearly surjective. To show it's also injective: Otherwise, assume $i < i'$ are 2 upper indices such that $\rho(i) = \rho(i') = k$. In particular, $|e_i| = |e_{i'}| = |e_k| + 1$. Then for some i-admissible x and i'-admissible element x' we have $k = \rho(i) = \rho(x)$ and $k = \rho(i') = \rho(x')$, that is, $dx = c_k e_k + \sum_{j>k} c_j e_j$ and $dx' = c'_k e_k + \sum_{j>k} c'_j e_j$ are both k-admissible, i.e. $c_k \neq 0, c'_k \neq 0$. If follows that $d(c'_k x - c_k x') = \sum_{j>k} *_j e_j$ and $c'_k x - c_k x'$ is still i-admissible. Hence, $\rho_d(c'_k x - c_k x') > k = \rho(i) = \max\{\rho_d(y) | y \text{ is } i\text{-admissible}\}$, contradiction.

Suppose $U = \{i_l, 1 \leq l \leq k | i_1 < i_2 < \ldots < i_k\}$ and $j_l := \rho(i_l), 1 \leq l \leq k$, then $L = \{j_l, 1 \leq l \leq k\}$. By definition of ρ, for each l there exists an i_l-admissible element, say e'_{i_l}, such that $e'_{j_l} := de'_{i_l}$ is j_l-admissible. We may even assume that $e'_{i_l} = e_{i_l} + \sum_{j>i_l} *_j e_j$. For each i in H, by definition, there exists an i-admissible element $e'_i = e_i + \sum_{j>i} *_j e_j$ such that $de'_i = 0$. We thus have constructed a set of elements $\{e'_1, e'_2, \ldots, e'_n\}$ in C with e'_i i-admissible, it follows that they form a basis of C. Define an automorphism φ of C by $\varphi(e'_i) = e_i$, and take $d_0 = \varphi \cdot d$. Then, $d_0 e_i = \varphi(de'_i)$. As a consequence, $d_0 e_{i_l} = e_{j_l}$ for $1 \leq l \leq k$ and $d_0 e_i = 0$ for $i \in H$. That is, d_0 is a Barannikov normal form of d.

Conversely, given a Barannikov normal form d_0 of d, there exist $2k$ distinct indices $i_1 < j_1, \ldots, i_k < j_k$ such that $d_0 e_{i_l} = e_{j_l}$ for $1 \leq l \leq k$ and $d_0 e_i = 0$ otherwise. Apply the definition of the 3 types of indices with respect to d_0, we must have $U = \{i_1, \ldots, i_k\}$ and $\rho(i_l) = \max\{\rho_{d_0}(x) | x \text{ is } i_l\text{-admissible}\} = j_l$, so $L = \{j_1, \ldots, j_k\}$. Hence, d_0 is uniquely determined by the partition $I = L \sqcup H \sqcup U$ and the bijection $\rho : U \xrightarrow{\sim} L$, which are determined by the isomorphism class of d. □

Definition 3.1.6. *Given a trivial Legendrian tangle (T, μ), a partition $I(T) = U \sqcup H \sqcup L$ together with a bijection $\rho : U \xrightarrow{\sim} L$ as in the proof of the previous lemma (see Equation (3.1.0.1)), will be called an m-*graded isomorphism type *of T, denoted by ρ for simplicity. Note: $\rho(i) > i$ and $|e_{\rho(i)}| = |e_i| - 1 (\text{mod} m)$ for all $i \in U$.*

Remark 3.1.7. *By Lemma 3.1.5, each m-graded isomorphism type ρ of T determines an unique isomorphism class $\mathcal{O}_m(\rho; k)$ of $\mathbb{Z}/m$-graded filtered k-complexes $(C(T), d)$. In other words, $\mathcal{O}_m(\rho; k)$ is the $B_m(T)$-orbit of the* canonical augmentation *ϵ_ρ (equivalently, the* Barannikov normal form *d_ρ determined by ρ), using the identification in Example 3.1.3. We thus obtain a decomposition of the augmentation variety for the trivial Legendrian tangle (T, μ):*

$$\text{Aug}_m(T; k) = \sqcup_\rho \mathcal{O}_m(\rho; k) \tag{3.1.0.2}$$

where ρ runs over all m-graded isomorphism types of T.

In addition, take a m-graded augmentation ϵ of $\mathcal{A}(T)$, or equivalently the m-graded filtered chain complex $C(\epsilon) = (C, d(\epsilon))$. Suppose ϵ is acyclic, *meaning that $(C, d(\epsilon))$ is acyclic or $H = \emptyset$ in the partition $I = L \sqcup H \sqcup U$ associated to $d(\epsilon)$. Then, the associated m-graded isomorphism type $\rho : U \xrightarrow{\sim} L$ can be identified with an m-*graded normal ruling *(denoted by the same ρ) of T.*

Remark 3.1.8. *In Lemma 3.1.5, given any complex* (C, d) *(or the corresponding augmentation* ϵ*), which determines a partition* $I = U \sqcup L \sqcup H$ *and a bijection* $\rho : U \xrightarrow{\sim} L$*, say* $U = \{i_1 < i_2 < \ldots < i_k\}$ *and* $\rho(i_l) = j_l$*. Then* $\varphi^{-1} \cdot d = d_0$ *is the Barannikov normal form for some* $\varphi \in Aut(C)$*. Can take the decomposition* $\varphi = D \circ \varphi_0$*, where* D *is diagonal and* φ_0 *is unipotent, i.e.* $\varphi_0(e_i) = e_i + \sum_{j>i} *_j e_j$*. Then* $(\varphi_0^{-1} \cdot d)(e_{i_l}) = c_l e_{j_l}$ *for* $c_l \in k^*$ *and* $1 \leq l \leq k$*, and* $(\varphi_0^{-1} \cdot d)(e_j) = 0$ *for the remaining cases. Such a complex* $(C, \varphi_0^{-1} \cdot d)$ *(or the corresponding augmentation* $\varphi_0^{-1} \cdot \epsilon$*) is called* standard*, and we say* $(C, \varphi_0^{-1} \cdot d)$ *is* standard with respect to ρ.

In fact, the unipotent automorphism φ_0 *can be taken to be canonical. See Lemma 3.3.7.*

Augmentation varieties for Legendrian tangles also satisfy a sheaf property, induced by the co-sheaf property of LCH DGAs in Section 2.2. More precisely, we have

Definition/Proposition 3.1.9. *Let* T *a Legendrian tangle in* J^1U*.*

(1) Let V *be an open subinterval of* U*, then the co-restriction of DGAs* $\iota_{UV} : \mathcal{A}(T|_V) \to \mathcal{A}(T)$ *induces a restriction* $r_{VU} = \iota_{VU}^* : \mathrm{Aug}_m(T; k) \to \mathrm{Aug}_m(T|_V; k)$*.*

(2) If $U = L \cup_V R$ *is the union of 2 open intervals* L, R *with non-empty intersection* V*, then the diagram of restriction maps*

$$\begin{array}{ccc} \mathrm{Aug}_m(T;k) & \xrightarrow{r_{RU}} & \mathrm{Aug}_m(T|_R;k) \\ {\scriptstyle r_{LU}}\downarrow & & \downarrow{\scriptstyle r_{VR}} \\ \mathrm{Aug}_m(T|_L;k) & \xrightarrow{r_{VL}} & \mathrm{Aug}_m(T|_V;k) \end{array} \tag{3.1.0.3}$$

gives a fiber product of augmentation varieties.

Take the left and right pieces of T, called T_L, T_R respectively. We get 2 restrictions of augmentation varieties $r_L = \iota_L^* : \mathrm{Aug}_m(T) \to \mathrm{Aug}_m(T_L)$ and $r_R = \iota_R^* : \mathrm{Aug}_m(T) \to \mathrm{Aug}_m(T_R)$. We can then define some subvarieties:

Definition 3.1.10. *Given* m*-graded isomorphism types* ρ_L, ρ_R *for* T_L, T_R *respectively, and* $\epsilon_L \in \mathcal{O}_m(\rho_L; k)$*. Define the varieties*

$$\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k) := \{\epsilon_L\} \times_{\mathrm{Aug}_m(T_L;k)} \times \mathrm{Aug}_m(T;k) \times_{\mathrm{Aug}_m(T_R;k)} \times \mathcal{O}_m(\rho_R;k)$$
$$\mathrm{Aug}_m(T, \rho_L, \rho_R; k) := \mathcal{O}_m(\rho_L;k) \times_{\mathrm{Aug}_m(T_L;k)} \times \mathrm{Aug}_m(T;k) \times_{\mathrm{Aug}_m(T_R;k)} \times \mathcal{O}_m(\rho_R;k)$$

$\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k)$ *will be called the* m-graded augmentation variety with boundary conditions (ϵ_L, ρ_R) *for* T*. When* $\epsilon_L = \epsilon_{\rho_L}$ *is the* canonical augmentation *of* T_L *corresponding to the Barannikov normal form determined by* ρ_L*, we will call* $\mathrm{Aug}_m(T, \epsilon_{\rho_L}, \rho_R; k)$ *the* m-graded augmentation variety (with boundary conditions (ρ_L, ρ_R)*) of* T*.*

By definition, we immediately obtain a decomposition of the full augmentation variety

$$\mathrm{Aug}_m(T;k) = \sqcup_{\rho_L,\rho_R}\mathrm{Aug}_m(T,\rho_L,\rho_R;k) \tag{3.1.0.4}$$

where ρ_L, ρ_R run over all m-graded isomorphism types of T_L, T_R respectively.

Note that the augmentation variety $\mathrm{Aug}_m(T,k)$ itself is not a Legendrian isotopy invariant. However, we can define

Definition 3.1.11. *Let $\mathbb{F}_q$ be any finite field, and ρ_L, ρ_R be m-*graded isomorphism types *of T_L, T_R respectively. The m-*graded augmentation number (with boundary conditions (ρ_L, ρ_R)) *of T over $\mathbb{F}_q$ is*

$$\mathrm{aug}_m(T,\rho_L,\rho_R;q) := q^{-\dim_{\mathbb{C}}\mathrm{Aug}_m(T,\epsilon_{\rho_L},\rho_R;\mathbb{C})}|\mathrm{Aug}_m(T,\epsilon_{\rho_L},\rho_R;\mathbb{F}_q)| \tag{3.1.0.5}$$

where $|\mathrm{Aug}_m(T,\epsilon_{\rho_L},\rho_R;\mathbb{F}_q)|$ is simply the counting of $\mathbb{F}_q$-points.

Remark 3.1.12. *Alternatively, we can use $\mathrm{Aug}_m(T,\rho_L,\rho_R;k)$ instead of $\mathrm{Aug}_m(T,\epsilon_{\rho_L},\rho_R;k)$ to define the augmentation number. However, this alternative definition only differs from the previous one by a normalized factor $q^{-\dim\mathcal{O}_m(\rho_L;k)}|\mathcal{O}_m(\rho_L;\mathbb{F}_q)| = (\frac{q-1}{q})^{|L|}$, where L comes from the partition $I(T_L) = U \sqcup H \sqcup L$ determined by ρ_L. See Corollary 3.3.8.*

In the next section, we will see that the augmentation numbers defined above are Legendrian isotopy invariants. However, for the purpose of clarity, we will now restrict ourselves to the case when ρ_L, ρ_R are m-graded normal rulings. In particular, this ensures that T has even left and even right endpoints.

3.2 Ruling polynomials compute augmentation numbers

Computation for augmentation numbers

Given a Legendrian tangle (T,μ). For the moment, we will assume T is placed with B base points so that *each right cusp is marked*. Label the crossings, cusps and base points away from the right cusps of T by $q_1,\ldots,q_n$ with x-coordinates, from left to right. Let $x_0 < x_1 < \ldots < x_n$ be the x-coordinates which cut T into elementary tangles. That is, x_0 and x_n are the the x-coordinates of the left and right end-points of T, and $x_{i-1} < x_{q_i} < x_i$ for all $1 \le i \le n$. Let $T_i = T|_{\{x_0<x<x_i\}}$ and $E_i := T|_{\{x_{i-1}<x<x_i\}}$ be the i-th elementary tangle around q_i, then $T = T_n = E_1 \circ E_2 \circ \ldots \circ E_n$ is the composition of n elementary tangles.

Fix m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively. Fix $\epsilon_L \in \mathcal{O}_m(\rho_L;k)$.

Definition 3.2.1. *For any m-graded normal ruling ρ of T such that $\rho|_{T_L} = \rho_L$ and $\rho|_{T_R} = \rho_R$, denote $\rho_i := \rho|_{(T_i)_R=(T_{i+1})_L}$ for $0 \le i \le n$. In particular, $\rho_0 = \rho_L, \rho_n = \rho_R$. Define the variety*

$$\begin{aligned}\mathrm{Aug}_m^{\rho}(T,\epsilon_L) &:= \mathrm{Aug}_m(E_1,\epsilon_L,\rho_1)\times_{\mathcal{O}_m(\rho_1)}\ldots\times_{\mathcal{O}_m(\rho_{n-1})}\mathrm{Aug}_m(E_n,\rho_{n-1},\rho_n)\\ \mathrm{Aug}_m^{\rho}(T,\rho_L) &:= \mathrm{Aug}_m(E_1,\rho_0,\rho_1)\times_{\mathcal{O}_m(\rho_1)}\ldots\times_{\mathcal{O}_m(\rho_{n-1})}\mathrm{Aug}_m(E_n,\rho_{n-1},\rho_n)\end{aligned}$$

while for simplicity we have ignored the coefficient field k.

Remark 3.2.2. *Given any elementary Legendrian tangle E: a single crossing, a left cusp, a (marked or unmarked) right cusp, or $2n$ parallel strands with a single base point, let ϵ be any m-graded augmentation of $\mathcal{A}(E)$ and denote $\epsilon_L := \epsilon|_{E_L}, \epsilon_R := \epsilon|_{E_R}$. If ϵ_L is acyclic (see Remark 3.1.7), then so is ϵ_R. By induction, this result then generalizes to all Legendrian tangles. For a justification, see Corollary 3.3.3.*

We then obtain a partition into subvarieties

$$\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k) = \sqcup_\rho \mathrm{Aug}^\rho_m(T, \epsilon_L; k) \tag{3.2.0.1}$$

where ρ runs over all m-graded normal rulings of T such that $\rho|_{T_L} = \rho_L$ and $\rho|_{T_R} = \rho_R$.

Consider the natural map

$$P_n : \mathrm{Aug}^\rho_m(T_n, \epsilon_L; k) \to \mathrm{Aug}_m^{\rho|_{T_{n-1}}}(T_{n-1}, \epsilon_L; k) \tag{3.2.0.2}$$

Clearly the fibers are $\mathrm{Aug}_m(E_n, \epsilon_{n-1}, \rho_n; k)$, where $\epsilon_{n-1} \in \mathcal{O}_m(\rho_{n-1}; k)$.

Lemma 3.2.3. *Let (E, μ) be an elementary Legendrian tangle: a single crossing q, a left cusp q, a marked right cusp q or $2n$ parallel strands with a single base point $*$. Let ρ be a m-graded normal ruling of E, denote $\rho_L := \rho|_{E_L}, \rho_R := \rho|_{E_R}$. Take any ϵ_L in $\mathcal{O}_m(\rho_L; k)$, have*

$$\mathrm{Aug}_m(E, \epsilon_L, \rho_R; k) \cong (k^*)^{-\chi(\rho)+B} \times k^{r(\rho)}$$

where B is the number of base points in E, $\chi(\rho) = s(\rho) - c_R$, c_R the number of right cusps in E. And, $s(\rho)$ and $r(\rho)$ are defined as in Definition 2.1.4.

We will not show the lemma until the Section 3.3.

Remark 3.2.4. *In fact, for any Legendrian tangle $T = T_n$ as above, one can show that*

$$\mathrm{Aug}^\rho_m(T, \epsilon_L; k) \cong (k^*)^{-\chi(\rho)+B} \times k^{r(\rho)}$$

See Theorem 3.3.10. However, for our purpose of points-counting, the previous lemma will suffice.

Assuming the lemma, we see that the map $P_n : \mathrm{Aug}^\rho_m(T_n, \epsilon_L; k) \to \mathrm{Aug}_m^{\rho|_{T_{n-1}}}(T_{n-1}, \epsilon_L; k)$ is surjective with smooth isomorphic fibers $(k^*)^{-\chi(\rho|_{E_n})+B(E_n)} \times k^{r(\rho|_{E_n})}$, where $B(E_n)$ denotes the number of base points in E_n. It follows that

$$\dim \mathrm{Aug}^\rho_m(T_n, \epsilon_L) = \dim \mathrm{Aug}_m^{\rho|_{T_{n-1}}}(T_{n-1}, \epsilon_L) - \chi(\rho|_{E_n}) + B(E_n) + r(\rho|_{E_n})$$
$$|\mathrm{Aug}^\rho_m(T_n, \epsilon_L; \mathbb{F}_q)| = |\mathrm{Aug}_m^{\rho|_{T_{n-1}}}(T_{n-1}, \epsilon_L; \mathbb{F}_q)|(q-1)^{-\chi(\rho|_{E_n})+B(E_n)} q^{r(\rho|_{E_n})}$$

So by induction, we obtain

$$\begin{aligned}&\dim Aug_m^{\rho}(T,\epsilon_L;k) = -\chi(\rho) + B - r(\rho) \\ &|\mathrm{Aug}_m^{\rho}(T,\epsilon_L;\mathbb{F}_q)| = (q-1)^{-\chi(\rho)+B} q^{r(\rho)}\end{aligned} \tag{3.2.0.3}$$

As a consequence of Equation (3.2.0.1), we then have

Lemma 3.2.5. *Given a Legendrian tangle (T,μ) with B base points so that each right cusp is marked, let ρ_L, ρ_R be m-graded normal rulings of T_L, T_R respectively, then for any $\epsilon_L \in \mathcal{O}_m(\rho_L;k)$, have*

$$\dim\mathrm{Aug}_m(T,\epsilon_L,\rho_R;k) = \max_{\rho}\{-\chi(\rho)+B+r(\rho)\} \tag{3.2.0.4}$$

and the augmentation number is given by

$$\mathrm{aug}_m(T,\rho_L,\rho_R;q) = q^{-\max_{\rho}\{-\chi(\rho)+B+r(\rho)\}} \sum_{\rho} (q-1)^{-\chi(\rho)+B} q^{r(\rho)} \tag{3.2.0.5}$$

where ρ runs over all m-graded normal rulings such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$.

Corollary 3.2.6 (Invariance of augmentation numbers)**.** *In the setting of the previous lemma with B fixed, then the augmentation numbers* $\mathrm{aug}_m(T,\rho_L,\rho_R;q)$ *are Legendrian isotopy invariants.*

Proof. Given a Legendrian isotopy $h: T \to T'$, by Lemma 2.1.9 there's a canonical bijection $\phi_h: \mathrm{NR}_T^m \xrightarrow{\sim} \mathrm{NR}_{T'}^m$ between the sets of m-graded normal rulings of T, T', which commutes with the restriction to left and right pieces, and $\chi(\phi_h(\rho)) = \chi(\rho)$ for any m-graded normal ruling of T. Moreover, by Remark 2.1.12, there's a constant C_r which only depend on T and h, such that $r(\phi_h(\rho)) = r(\rho) + C_r$. Apply the previous lemma, we get

$$\begin{aligned}\dim\mathrm{Aug}_m(T',\rho_L,\rho_R;k) &= \max_{\rho}\{-\chi(\phi_h(\rho))+B+r(\phi_h(\rho))\} \\ &= \max_{\rho}\{-\chi(\rho)+B+r(\rho)\}+C_r \\ &= \dim\mathrm{Aug}_m(T,\rho_L,\rho_R;k)+C_r\end{aligned}$$

where ρ runs over all m-graded normal rulings of T such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$, and

$$\begin{aligned}\mathrm{aug}_m(T',\rho_L,\rho_R;q) &= q^{-\dim\mathrm{Aug}_m(T,\rho_L,\rho_R;k)-C_r} \sum_{\rho} (q-1)^{-\chi(\phi_h(\rho))+B} q^{r(\phi_h(\rho))} \\ &= q^{-\dim\mathrm{Aug}_m(T,\rho_L,\rho_R;k)-C_r} \sum_{\rho} (q-1)^{-\chi(\rho)+B} q^{r(\rho)+C_r} \\ &= \mathrm{aug}_m(T,\rho_L,\rho_R;q)\end{aligned}$$

where ρ runs as above. □

Ruling polynomials compute augmentation numbers

Theorem 3.2.7. *Let T be a Legendrian tangle equipped with a $\mathbb{Z}/2r$-valued Maslov potential μ and B base points so that each connected component containing a right cusp has at least one base point. Fix a nonnegative integer m dividing $2r$ and m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively, then the augmentation numbers and Ruling polynomials of (T, μ) are related by*

$$\mathrm{aug}_m(T, \rho_L, \rho_R; q) = q^{-\frac{d+B}{2}} z^B < \rho_L | R_T^m(z) | \rho_R > \qquad (3.2.0.6)$$

where q is the order of a finite field $\mathbb{F}_q$, $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, d is the maximal degree in z of $< \rho_L | R_T^m(z) | \rho_R >$.

Proof. Firstly, we prove the theorem when *each right cusp is marked* in T. We need the following direct generalization of [15, lem.3.5]

Lemma 3.2.8. *Let (T, μ) be any Legendrian tangle and fix m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively. Let ρ and ρ' be any two m-graded normal rulings of T which restricts to ρ_L (resp. ρ_R) on T_L (resp. T_R), then*

$$-\chi(\rho) + 2r(\rho) = -\chi(\rho') + 2r(\rho')$$

Note: Unlike [15, lem.3.5], we do not assume T to have nearly plat front diagram. We will postpone the proof of the lemma until the end of this subsection.

Assuming Lemma 3.2.8, we prove Theorem 3.2.7. Fix ρ_0 such that $\mathrm{dimAug}_m(T, \rho_L, \rho_R) = -\chi(\rho_0) + B + r(\rho_0)$. It follows from lemma 3.2.8 that $-\chi(\rho_0)$ is also maximal, hence $d = -\chi(\rho_0) = \mathrm{max.deg}_z < \rho_L | R_T^m(z) | \rho_R >$. For any m-graded normal ruling ρ, Lemma 3.2.8 implies that $r(\rho) - r(\rho_0) = \frac{1}{2}(d + \chi(\rho))$. Plug this into equation (3.2.0.5), we obtain

$$\begin{aligned} aug_m(T, \rho_L, \rho_R; q) &= q^{-d-B-r(\rho_0)} \sum\nolimits_\rho (q-1)^{-\chi(\rho)+B} q^{r(\rho)} \\ &= q^{-\frac{d+B}{2}} \sum\nolimits_\rho (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-\chi(\rho)+B} \\ &= q^{-\frac{d+B}{2}} z^B < \rho_L | R_T^m(z) | \rho_R > \end{aligned}$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ and ρ runs over all m-graded normal rulings of T such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$.

In general, the theorem reduces to the previous case via Lemma 3.2.9 below. □

Lemma 3.2.9 (Dependence on the base points of augmentation numbers)**.** *As in the previous theorem, let (T, μ) be a Legendrian tangle with B base points $*_1, \ldots, *_B$ so that each connected component containing a right cusp has at least one base point. Fix m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively, then the* normalized augmentation number

$$\mathrm{N.aug}_m(T, \rho_L, \rho_R; q) := q^{\frac{d+B}{2}} z^{-B} \mathrm{aug}_m(T, \rho_L, \rho_R; q)$$

is independent of the number and positions of the base points on T.

Proof. To express the explicit dependence on the base points, we write $\mathrm{aug}_m(T, \rho_L, \rho_R; q) = \mathrm{aug}_m(T, *_1, \ldots, *_B, \rho_L, \rho_R; q)$ and $\mathrm{N.aug}_m(T, \rho_L, \rho_R; q) = \mathrm{N.aug}_m(T, *_1, \ldots, *_B, \rho_L, \rho_R; q)$.

Firstly, we show that the (normalized) augmentation number is *independent of the positions of the base points* in each connected component of T. It suffices to show that: Let $(*_1, \ldots, *_B)$ and $(*'_1, \ldots, *'_B)$ be 2 collections of base points on T, which are identical except that for some i, when $*'_i$ is the result of sliding $*_i$ across a crossing of $Res(T)$. Then $\mathrm{aug}_m(T, *_1, \ldots, *_B, \rho_L, \rho_R; q) = \mathrm{aug}_m(T, *'_1, \ldots, *'_B, \rho_L, \rho_R; q)$. Notice that a base point on a right cusp corresponds to a base point on the boundary of the "invisible" disk after resolution.

Suppose $*_i, *'_i$ lie on the opposite sides of a in $Res(T)$ and the orientation of T goes from $*_i$ to $*'_i$, where a is a crossing or right cusp of T. We firstly assume the strand containing $*_i, *'_i$ is the over-strand at a of $Res(T)$. If u is an *admissible disk* as in Definition 2.2.2 with an initial vertex at a, and $w(u), w'(u)$ are the weights of u in the DGAs $(\mathcal{A}(T, *_1, \ldots, *_B), \partial), (\mathcal{A}(T, *'_1, \ldots, *'_B), \partial')$ respectively. Then $w(u) = t_i^{-1} w'(u)$, i.e. $\partial'(t_i^{-1}a) = \partial a$. If u is an admissible disk with at least one negative vertex at a, then $w'(u)$ is the result of replacing each a by $t_i^{-1}a$ in $w(u)$. In other words, we have an isomorphism of $\mathbb{Z}/2r$-graded DGAs $\phi : \mathcal{A}(T, *_1, \ldots, *_B) \xrightarrow{\sim} \mathcal{A}(T, *'_1, \ldots, *'_B)$ given by $\phi(a) = t_i^{-1}a, \phi(a') = a'$ for all generators $a' \neq a$, and $\phi(t_j) = t_j$. It follows that ϕ induces an isomorphism $\phi^* : \mathrm{Aug}_m(T, *'_1, \ldots, *'_B, \epsilon_{\rho_L}, \rho_R; k) \xrightarrow{\sim} \mathrm{Aug}_m(T, *_1, \ldots, *_B, \epsilon_{\rho_L}, \rho_R; k)$ defined by $\phi^* \epsilon' = \epsilon' \circ \phi$. Notice that ϕ' only changes the values of augmentations at a, the boundary condition $(\epsilon_{\rho_L}, \rho_R)$ is indeed preserved by ϕ^*. Now, by definition $\mathrm{aug}_m(T, *_1, \ldots, *_B, \rho_L, \rho_R; q) = \mathrm{aug}_m(T, *'_1, \ldots, *'_B, \rho_L, \rho_R; q)$.

If the strand containing $*_i, *'_i$ is the under-strand at a of $Res(T)$. A similar argument shows that $\phi : \mathcal{A}(T, *_1, \ldots, *_B) \xrightarrow{\sim} \mathcal{A}(T, *'_1, \ldots, *'_B)$, given by $\phi(a) = at_i, \phi(a') = a'$ for $a' \neq a$ and $\phi(t_j) = t_j$, defines an isomorphism of $\mathbb{Z}/2r$-graded DGAs. Again, the desired equality follows as in the previous case.

Secondly, we show that the normalized augmentation number is *independent of the number of base points* on T. By the first half of the result proved above, it suffices to show that: Let $*_1, \ldots, *_B, *_{B+1}$ a collection of base points on T such that $*_B, *_{B+1}$ lie in a small neighborhood of T avoiding the crossings, cusps and other base points, then $\mathrm{N.aug}_m(T, *_1, \ldots, *_B, \rho_L, \rho_R; q) = \mathrm{N.aug}_m(T, *_1, \ldots, *_B, *_{B+1}, \rho_L, \rho_R; q)$, or equivalently,

$$\mathrm{aug}_m(T, *_1, \ldots, *_{B+1}, \rho_L, \rho_R; q) = \frac{(q-1)}{q} \mathrm{aug}_m(T, *_1, \ldots, *_B, \rho_L, \rho_R; q) \qquad (3.2.0.7)$$

In this case, there's a natural morphism of $\mathbb{Z}/2r$-graded DGAs $\phi : \mathcal{A}(T, *_1, \ldots, *_B) \rightarrow \mathcal{A}(T, *_1, \ldots, *_{B+1})$ given by $\phi(a) = a$ for all generators a, $\phi(t_i) = t_i$ for $i < B$ and $\phi(t_B) = t_B t_{B+1}$. Indeed, we obtain an isomorphism of DGAs $\Phi : \mathcal{A}(T, *_1, \ldots, *_B)[t, t^{-1}] \xrightarrow{\sim} \mathcal{A}(T, *_1, \ldots, *_{B+1})$ given by $\Phi(a) = a, \phi(t_i) = t_i, i < B, \Phi(t_B) = t_B t_{B+1}$ and $\Phi(t) = t_{B+1}$. Hence, we obtain an induced isomorphism

$$\Phi^* : \mathrm{Aug}_m(T, *_1, \ldots, *_{B+1}, \epsilon_{\rho_L}, \rho_R; k) \xrightarrow{\sim} \mathrm{Aug}_m(T, *_1, \ldots, *_B, \epsilon_{\rho_L}, \rho_R; k) \times k^*$$

given by $\Phi^* := \phi^* \times e_{B+1}$, with $\phi^*\epsilon(a) = \epsilon(a)$ for all generators a, $\phi^*\epsilon(t_i) = \epsilon(t_i)$ for $i < B$ and $\phi^*\epsilon(t_B) = \epsilon(t_B)\epsilon(t_{B+1})$, and $e_{B+1}(\epsilon) = \epsilon(t_{B+1})$. By definition of augmentation numbers, it then follows that the equality (3.2.0.7) holds. □

Now, let's prove Lemma 3.2.8. For any m-graded normal ruling ρ of (T, μ), *define* $r'(\rho)$ to be the number of m-graded returns of ρ. It suffices to show $-\chi(\rho)+2r'(\rho) = -\chi(\rho')+2r'(\rho')$ for any ρ, ρ' as in Lemma 3.2.8. However, $-\chi(\rho) = s(\rho) - c_R$ implies

$$\begin{aligned} -\chi(\rho) + 2r'(\rho) &= (s(\rho) + r'(\rho) + d(\rho)) - c_R + r'(\rho) - d(\rho) \\ &= r_m - c_R + r'(\rho) - d(\rho) \end{aligned}$$

where r_m is the number of crossings of the front T of degree 0 modulo m. Hence, Lemma 3.2.8 is a consequence of the following

Proposition 3.2.10. *Let (T, μ) be any Legendrian tangle and fix m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively. Then for any m-graded normal ruling ρ such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$, $r'(\rho) - d(\rho)$ is independent of ρ.*

Before the proof, let's firstly make some definitions. For any m-graded isomorphism type (Definition 3.1.6) ρ of a trivial Legendrian tangle E of n parallel strands. So ρ determines a partition $I = U \sqcup L \sqcup H$ and a bijection $\rho : U \xrightarrow{\sim} L$, where $I = I(E) = \{1, 2, \ldots, n\}$ is the set of left endpoints of E. Notice that $H = \emptyset$ when ρ is a m-graded normal ruling (Remark 3.1.7). For each i in H, we take $\rho(i) := \infty$. Now, we define the subsets $I(i), i \in I$, $A(i) = A_\rho(i), i \in U \sqcup H$ of I, and an index $A(\rho)$, depending on ρ as follows:

Definition 3.2.11. *For any $i \in I$, define*

$$I(i) := \{j \in I | j > i, \mu(j) = \mu(i)(\text{mod} m)\}.$$

Note: $I(i)$ is independent of ρ. Now for any $i \in U \sqcup H$, define

$$A(i) = A_\rho(i) := \{j \in U \sqcup H | j \in I(i) \text{ and } \rho(j) < \rho(i)\}.$$

Note: for any $j \in A(i)$, have $\rho(j) < \rho(i) \leq \infty$, hence we necessarily have $j \in U$. Now, define $A(\rho) \in \mathbb{N}$ by

$$A(\rho) := \sum_{i \in U \sqcup H} |A(i)| + \sum_{i \in L} |I(i)|.$$

See Corollary 3.3.8 for an interpretation of $A(\rho)$.

With the definition above, we can now prove the proposition.

Proof of Proposition 3.2.10. Assume T lives over the interval $[x_0, x_1]$. Let ρ be any m-graded normal ruling of T such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$. For each x in $[x_0, x_1]$ avoiding the crossings and cusps of T, define $A(x) := A(\rho|_{\{x\}})$. In particular, $A(x_0) = A(\rho_L)$ and $A(x_1) = A(\rho_R)$.

Observe that, as x increases, $A(x)$ increases (resp. decreases) by 1 when passing an m-graded return (resp. m-graded departure) of ρ and is unchanged when passing a crossing of all other types. When passing a right cusp q, let x_c, x'_c be the x-coordinate immediately before and after q. Suppose $\rho_c := \rho|_{x=x_c}$ and $\rho'_c := \rho|_{x=x'_c}$ determine the partitions $I_c = I(T|_{x=x_c}) = U_c \sqcup L_c$ and $I'_c = I(T|_{x=x'_c}) = U'_c \sqcup L'_c$ respectively. Suppose q connects strands $k, k+1$ of $T|_{x=x_c}$, then $k \in U_c, k+1 \in L_c$, $\rho_c(k) = k+1$ and $A_{\rho_c}(k) = \emptyset$. Denote $I_c^a := \{i \in I(T|_{x=x_c}) | \mu(i) = a(\text{mod} m)\}$, $U_c^a := \{i \in U_c | \mu(i) = a(\text{mod} m)\}$ and $L_c^a := \{i \in L_c | \mu(i) = a(\text{mod} m)\}$ for all congruence classes $a(\text{mod} m)$. Say, $\mu(k) = a(\text{mod} m)$. It follows that

$$\begin{aligned} & A(x_c) - A(x'_c) - |I_c(k+1)| \\ = & \sum_{i \in U_c, k \in A_{\rho_c}(i)} 1 + \sum_{i \in L_c, k \in I_c(i)} 1 + \sum_{i \in L_c, k+1 \in I_c(i)} 1 \\ = & \sum_{i \in U_c^a, i<k, \rho_c(i)>k+1} 1 + \sum_{i \in L_c^a, i<k} 1 + \sum_{j=\rho_c^{-1}(i) \in U_c^a, j<\rho_c(j)=i<k+1} 1 \\ = & \sum_{i \in U_c^a, i<k} 1 + \sum_{i \in L_c^a, i<k} 1 \\ = & |\{i \in I_c^a | i < k\}|. \end{aligned}$$

Hence,

$$\begin{aligned} A(x_c) - A(x'_c) &= |\{i \in I_c | i > k+1, \mu(i) = \mu(k+1)(\text{mod} m)\}| \\ &+ |\{i \in I_c | i < k, \mu(i) = \mu(k)(\text{mod} m)\}| \end{aligned}$$

is independent of ρ. Similarly, when passing a left cusp, $A(x)$ only changes by a constant, which only depends on (T, μ) near the cusp, not on ρ.

As a consequence, by moving x from x_0 to x_1, we obtain that $A(\rho_R) - A(\rho_L) = A(x_1) - A(x_0) = r'(\rho) - d(\rho) + C$ for some constant C, which depends only on (T, μ), not on ρ. It follows that $r'(\rho) - d(\rho)$ is independent of ρ. □

Remark 3.2.12. *In [33], we have also considered the concepts of* m-graded generalized normal rulings *and* m-graded generalized Ruling polynomials *for Legendrian tangles. Moreover, the previous main results admit a direct generalization to this setting, and essentially the same arguments apply. See also [21] for some applications of generalized normal rulings to the study of Legendrian knots in* J^1S^1.

Example

Example 3.2.13. *Consider the Legendrian tangle* (T, μ) *in Example 2.1.13 (See Figure 2.1.2 (left)), with no base point. Hence,* $B = 0$. *Let's check Theorem 3.2.7 with our example by a* *direct calculation.*

Let $b_{ij}, 1 \leq i < j \leq 4$ *be the pairs of right end points of* T, *so* $\mathcal{A}(T_R)$ *is generated by* b_{ij}*'s with the grading:* $|b_{12}| = |b_{13}| = |b_{24}| = |b_{34}| = 0, |b_{23}| = -1, |b_{14}| = 1$. *By Example 2.1.13,* T_R *has 2 m-graded normal rulings* $(\rho_R)_1, (\rho_R)_2$. *Let's firstly determine the orbits* $\mathcal{O}_m((\rho_R)_1;k), \mathcal{O}_m((\rho_R)_2;k)$. *Use the identification in Example 3.1.3, given any m-graded augmentation* ϵ_R *for* T_R, *denote by* d_R *the corresponding differential for* $C(T_R)$. *Let* $I := \{1,2,3,4\}$ *be the set of right endpoints of* T.

By the proof of Lemma 3.1.5, $d_R \in \mathcal{O}_m((\rho_R)_1;k)$ *if and only if the partition and bijection determined by* d_R *is* $I = U \sqcup L$ *and* $(\rho_R)_1 : U \xrightarrow{\sim} L$, *where* $U = \{1,3\}, L = \{2,4\}$ *and* $(\rho_R)_1(1) = 2, (\rho_R)_1(3) = 4$. *That is, the condition says* $1,3$ *are* d_R*-upper, and* $(\rho_R)_1(i) = \rho_{d_R}(i) := \max\{\rho_{d_R}(x) | x \text{ is } i\text{-admissible}\}$ *for* $i = 1,3$, *equivalently,* $< d_R e_1, e_2 > \neq 0$ *and* $< d_R e_3, e_4 > \neq 0$. *Hence, we have*

$$\mathcal{O}_m((\rho_R)_1;k) = \{\epsilon_R \in \mathrm{Aug}_m(T_R;k) | \epsilon(b_{12}) \neq 0, \epsilon(b_{34}) \neq 0\}.$$

Similarly, $d_R \in \mathcal{O}_m((\rho_R)_2;k)$ *if and only if* $1,2$ *are* d_R*-upper and* $(\rho_R)_2(i) = \rho_{d_R}(i) = \max\{\rho_{d_R}(x) | x \text{ is } i\text{-admissible}\}$ *for* $i = 1,2$. *For* $i = 1$, *the previous condition says* $< d_R e_1, e_2 >= 0$ *(otherwise,* $\rho_{d_R}(1) = 2$, *contradiction),* $< d_R e_2, e_3 >= 0$ *(Otherwise, 2 is* d_R*-upper and* $\rho_{d_R}(2) = 3$, *contradiction), and* $< d_R e_1, e_3 > \neq 0$*; For* $i = 2$, *the previous condition says* $< d_R e_2, e_4 > \neq 0$ *and* $< d_R e_3, e_4 >= 0$. *As a consequence, we have*

$$\mathcal{O}_m((\rho_R)_2;k) = \{\epsilon_R \in \mathrm{Aug}_m(T_R;k) | \epsilon_R(b_{12}), \epsilon_R(b_{23}), \epsilon_R(b_{34}) = 0, \epsilon_R(b_{13}), \epsilon_R(b_{24}) \neq 0\}.$$

Now, let ϵ *be any m-graded k-augmentation of* T, *denote by* ϵ_L, ϵ_R *the restriction of* ϵ *to* T_L, T_R *respectively. Let* $x_i = \epsilon(a_i)$, *and* $x_{ij} = \epsilon(a_{ij})$ *for* $i < j$. *Notice that* $\epsilon(a_{23}) = 0 = \epsilon(a_{14})$. *By Example 2.2.8, the full augmentation variety for* T *is:*

$$\mathrm{Aug}_m(T;k) = \{(x_i, x_{ij}) | x_{23}, x_{14} = 0, \sum_{i<k<j} (-1)^{|a_{ik}|+1} x_{ik} x_{kj} = 0\}$$

for $m \neq 1$ *and*

$$\mathrm{Aug}_m(T;k) = \{(x_i, x_{ij}) | x_{23} = 0, \sum_{i<k<j} (-1)^{|a_{ik}|+1} x_{ik} x_{kj} = 0\}$$

for $m = 1$. *Moreover, the co-restriction* $\iota_R : \mathcal{A}(T_R) \to \mathcal{A}(T)$ *is given by*

$$\begin{aligned}
\iota_R(b_{12}) &= a_{13}(1 + a_2 a_3) + a_{12}(a_1 + a_3 + a_1 a_2 a_3);\\
\iota_R(b_{13}) &= a_{12}(1 + a_1 a_2) + a_{13} a_2;\\
\iota_R(b_{14}) &= a_{14}; \iota_R(b_{23}) = 0;\\
\iota_R(b_{24}) &= (1 + a_2 a_1) a_{34} - a_2 a_{24};\\
\iota_R(b_{34}) &= (1 + a_3 a_2) a_{24} - (a_1 + a_3 + a_3 a_2 a_1) a_{34}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\epsilon_R(b_{12}) &= x_{13}(1+x_2x_3)+x_{12}(x_1+x_3+x_1x_2x_3);\\
\epsilon_R(b_{13}) &= x_{12}(1+x_1x_2)+x_{13}x_2;\\
\epsilon_R(b_{14}) &= x_{14}; \epsilon_R(b_{23}) = 0;\\
\epsilon_R(b_{24}) &= (1+x_1x_2)x_{34}-x_2x_{24};\\
\epsilon_R(b_{34}) &= (1+x_2x_3)x_{24}-(x_1+x_3+x_1x_2x_3)x_{34}.
\end{aligned}$$

With the preparation above, we have the following augmentation variety and augmentation number (with fixed boundary conditions) associated to (T,μ), corresponding to each case in Example 2.1.13:
(1). *Notice that for $\epsilon_L = \epsilon_{(\rho_L)_1}$, have $x_{12} = 1 = x_{34}$ and $x_{ij} = 0$ otherwise. Hence, for the boundary conditions $((\rho_L)_1, (\rho_R)_1)$ (see Definition 3.1.10), have*

$$\begin{aligned}
&\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1}),(\rho_R)_1;k) = \{\epsilon\in \mathrm{Aug}_m(T;k)|\epsilon_L = \epsilon_{(\rho_L)_1}, \epsilon_R\in\mathcal{O}_m((\rho_R)_1;k)\}\\
=\ & \{(x_i)_{1\leq i\leq 3}\in k^3 | x_1+x_3+x_1x_2x_3\neq 0\}\\
=\ & k^*\times k\sqcup k^*\times k\sqcup (k^*)^3
\end{aligned}$$

where in the decomposition of the last equality, the subvarieties are $\{x_1 = 0, x_3\neq 0\}$, $\{x_3 = 0, x_1\neq 0\}$ and $\{x_1\neq 0, x_3\neq 0, x_1+x_3+x_1x_2x_3\neq 0\}$ respectively. Hence, by Definition 3.1.11 and Example 2.1.13, the augmentation number is

$$\begin{aligned}
&\mathrm{aug}_m(T,(\rho_L)_1,(\rho_R)_1;q) = q^{-3}(2(q-1)q+(q-1)^3)\\
=\ & q^{-\frac{3}{2}}(2z+z^3) = q^{-\frac{d}{2}} < (\rho_L)_1|R_T^m(z)|(\rho_R)_1 >
\end{aligned}$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ and $3 = d = \max.\deg_z < (\rho_L)_1|R_T^m(z)|(\rho_R)_1 >$.
(2). *For the boundary conditions $((\rho_L)_1, (\rho_R)_2)$, have*

$$\begin{aligned}
&\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1}),(\rho_R)_2;k) = \{\epsilon\in \mathrm{Aug}_m(T;k)|\epsilon_L = \epsilon_{(\rho_L)_1}, \epsilon_R\in\mathcal{O}_m((\rho_R)_2;k)\}\\
=\ & \{\{(x_i)_{1\leq i\leq 3}\in k^3 | x_1+x_3+x_1x_2x_3 = 0, 1+x_1x_2\neq 0\}\\
=\ & \{(x_1,x_2)\in k^2|1+x_1x_2\neq 0\} = k\sqcup (k^*)^2
\end{aligned}$$

where in the decomposition of the last equality, the subvarieties are $\{x_1 = 0, x_2\in k\}$ and $\{x_1\neq 0, 1+x_1x_2\neq 0\}$ respectively. Hence, by Definition 3.1.11 and Example 2.1.13, the augmentation number is:

$$\begin{aligned}
&\mathrm{aug}_m(T,(\rho_L)_1,(\rho_R)_2;q) = q^{-2}(q+(q-1)^2)\\
=\ & q^{-1}(1+z^2) = q^{-\frac{d}{2}} < (\rho_L)_1|R_T^m(z)|(\rho_R)_2 >
\end{aligned}$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ and $2 = d = \max.\deg_z < (\rho_L)_1|R_T^m(z)|(\rho_R)_2 >$.

(3). *Notice that for* $\epsilon_L = \epsilon_{(\rho_L)_2}$*, have* $x_{13} = 1 = x_{24}$ *and* $x_{ij} = 0$ *otherwise. Hence, for the boundary conditions* $((\rho_L)_2, (\rho_R)_1)$*, have*

$$\begin{aligned} & \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}), (\rho_R)_1; k) = \{\epsilon \in \mathrm{Aug}_m(T; k) | \epsilon_L = \epsilon_{(\rho_L)_2}, \epsilon_R \in \mathcal{O}_m((\rho_R)_1; k)\} \\ = \ & \{\{(x_i)_{1\leq i\leq 3} \in k^3 | 1 + x_2x_3 \neq 0\} \\ = \ & k^2 \sqcup k \times (k^*)^2 \end{aligned}$$

where in the decomposition of the last equality, the subvarieties are $\{x_2 = 0, (x_1, x_3) \in k^2\}$ *and* $\{x_2 \neq 0, 1 + x_2x_3 \neq 0, x_1 \in k\}$ *respectively. Hence, by Definition 3.1.11 and Example 2.1.13, the augmentation number is:*

$$\begin{aligned} & \mathrm{aug}_m(T, (\rho_L)_1, (\rho_R)_2; q) = q^{-3}(q^2 + q(q-1)^2) \\ = \ & q^{-1}(1 + z^2) = q^{-\frac{d}{2}} < (\rho_L)_2 | R_T^m(z) | (\rho_R)_1 > \end{aligned}$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ *and* $2 = d = \mathrm{max.deg}_z < (\rho_L)_2 | R_T^m(z) | (\rho_R)_1 >$.
(4). *For the boundary conditions* $((\rho_L)_2, (\rho_R)_2)$*, have*

$$\begin{aligned} & \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}), (\rho_R)_2; k) = \{\epsilon \in \mathrm{Aug}_m(T; k) | \epsilon_L = \epsilon_{(\rho_L)_2}, \epsilon_R \in \mathcal{O}_m((\rho_R)_2; k)\} \\ = \ & \{\{(x_i)_{1\leq i\leq 3} \in k^3 | 1 + x_2x_3 = 0, x_2 \neq 0\} \\ = \ & k \times k^* \end{aligned}$$

Hence, by Definition 3.1.11 and Example 2.1.13, the augmentation number is:

$$\begin{aligned} & \mathrm{aug}_m(T, (\rho_L)_2, (\rho_R)_2; q) = q^{-2}q(q-1) \\ = \ & q^{-\frac{1}{2}}z = q^{-\frac{d}{2}} < (\rho_L)_2 | R_T^m(z) | (\rho_R)_2 > \end{aligned}$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ *and* $1 = d = \mathrm{max.deg}_z < (\rho_L)_2 | R_T^m(z) | (\rho_R)_2 >$.

Altogether, the calculation matches with Theorem 3.2.7 in each case.

3.3 Augmentation varieties for elementary Legendrian tangles

The main goal of this section is to show Lemma 3.2.3. More generally, we also obtain a ruling decomposition (a finer structure) for the augmentation varieties $\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k)$ (see Theorem 3.3.10).

The identification between augmentations and A-form MCSs

Let (E, μ) be any elementary Legendrian tangle: a single crossing q, a left cusp q, a (marked or unmarked) right cusp q, or n parallel strands with a single base point q. Assume E has n_L left endpoints and n_R right endpoints, and denote $\mu_L := \mu|_{E_L}, \mu_R := \mu|_{E_R}$.

Let ϵ be any m-graded k-augmentation of $\mathcal{A}(E)$. Denote $\epsilon_L := \epsilon|_{E_L}, \epsilon_R = \epsilon|_{E_R}$, where ϵ_R is induced from ϵ via $\iota_R : \mathcal{A}(E_R) \to \mathcal{A}(E)$. By Example 3.1.3, we can identify ϵ_L and ϵ_R with some $\mathbb{Z}/m$-graded filtered complexes $(C(E_L), d_L)$ and $(C(E_R), d_R)$ respectively. We know d_R is completely determined by d_L and the information of ϵ near q. To make this precise, we firstly introduce the following

Definition 3.3.1. *A* handleslide *is a* vertical line segment H_r *lying on two strands of* (T, μ), *equipped with a* coefficient $r \in k$, *where* (T, μ) *is some trivial Legendrian tangle of* n *parallel strands. For simplicity, we denote such a handleslide by* H_r. *A handleslide* H_r *is* m-graded *if either* $r = 0$ *or its end-points belong to 2 strands having the same Maslov potential value modulo* m.

A $\mathbb{Z}/m$-graded handleslide H_r with coefficient r between strands $j < k$, is also equivalent to an $\mathbb{Z}/m$-graded filtered *elementary transformation* $H_r : C((H_r)_L) \xrightarrow{\sim} C((H_r)_R)$ (closely related to Morse complex sequences (MCSs) in [15]):

$$H_r(e_i) = \begin{cases} e_i & i \neq j \\ e_j - re_k & i = j \end{cases}$$

Now, by a direct calculation we have

Lemma 3.3.2. *Given* (E, μ) *and* ϵ *as above.*

1. *If* E *is a single crossing* q *between strands* $k, k+1$. *Then there's an isomorphism of* Z/m*-graded (not necessarily filtered) complexes* $\varphi : (C(E_L), d_L) \xrightarrow{\sim} (C(E_R), d_R)$ *given by* $\varphi = s_k \circ H_r$ *for* $r = -\epsilon(q)$, *where* $s_k : C(E_L) \xrightarrow{\sim} C(E_R)$ *is the* $\mathbb{Z}/m$*-graded* elementary transformation

   $$s_k(e_i) = \begin{cases} e_i & i \neq k, k+1 \\ e_{k+1} & i = k \\ e_k & i = k+1 \end{cases}$$

   and $H_r : C(E_L) \xrightarrow{\sim} C(E_L)$ *is the handleslide between strands* $k, k+1$ *of* E_L. *Note:* $< d_L e_k, e_{k+1} >= 0 =< d_R e_k, e_{k+1} >$.
   Pictorially, we can represent s_k *by the front diagram* E *with a crossing between strands* $k, k+1$, *hence* φ *is represented by the front diagram* E *with a handleslide of coefficient* r *between strands* $k, k+1$ *to the left of* q.

2. *If* E *is a left cusp* q *connecting strands* $k, k+1$ *of* E_R. *Then as a* $\mathbb{Z}/m$*-graded complex,* $(C(E_R), d_R)$ *is a direct sum of* $(C(E_L), d_L)$ *and the acyclic complex* (Span$\{e_k, e_{k+1}\}$, $d_R e_k = (-1)^{\mu_R(k)} e_{k+1}$), *via the morphism* $\varphi : (C(E_L), d_L) \hookrightarrow (C(E_R), d_R)$:

   $$\varphi(e_i) = \begin{cases} e_i & i < k \\ e_{i+2} & i \geq k \end{cases}$$

   Pictorially, we can simply represent φ *by the front diagram* E.

3. *If E is a right cusp q connecting strands $k, k+1$ of E_L. Let t be the generator corresponding to the base point in $\mathcal{A}(E)$ if q is marked, and 1 otherwise. Then there's an morphism of complexes $\varphi : (C(E_L), d_L) \to (C(E_R), d_R)$ given by $\varphi = \varphi_c \circ Q \circ H_r$, where $H_r : (C(E_L), d_L) \xrightarrow{\sim} (C(E_L), d'_L)$ is the handleslide with coefficient $r = -\epsilon(q)$ between strands $k, k+1$ of E_L, $Q : (C(E_L), d'_L) \to (C(E_L), d'_L)/\mathrm{Span}\{e_k, d'_L e_k\}$ is the natural quotient map, and $\varphi_c : (C(E_L), d'_L)/\mathrm{Span}\{e_k, d'_L e_k\} \xrightarrow{\sim} (C(E_R), d_R)$ is the isomorphism defined by*

$$\varphi_c([e_i]) = \begin{cases} e_i & i < k \\ e_{i-2} & i > k+1 \end{cases}$$

Note: $< d_L e_k, e_{k+1} >=< d'_L e_k, e_{k+1} >= (-1)^{\mu_L(k)}(-\epsilon(t)^{\sigma(q)})$ (see Definition 2.2.12 for $\sigma(q)$), this ensures that the quotient $(C(E_L), d'_L)/\mathrm{Span}\{e_k, d'_L e_k\}$ is freely generated by $[e_i], i \neq k, k+1$ as a k-module.
Pictorially, we can represent *$\varphi_c \circ Q$ by the front E (with coefficient $\epsilon(t)^{\sigma(q)}$ attached to the base point if q is marked), then φ is represented by the front E with a handleslide between strands $k, k+1$ of E_L to the left of q.*

4. *If E is a single base point q on the strand k. Let $\lambda := \epsilon(q)$ (resp. $\epsilon(q)^{-1}$) if the orientation of the strand k is right moving (resp. left moving). Then there's an isomorphism of complexes $\varphi : (C(E_L), d_L) \to (C(E_R), d_R)$ via*

$$\varphi(e_i) = \begin{cases} e_i & i \neq k \\ \lambda e_k & i = k \end{cases}$$

Pictorially, we can simply represent *φ by the front E with the coefficient λ attached to the base point.*

Corollary 3.3.3. *There's an isomorphism $H_*(C(E_L), d_L) \xrightarrow{\sim} H_*(C(E_R), d_R)$ of $\mathbb{Z}/m$-graded k-modules . In particular, if ϵ_L is acyclic, then so is ϵ_R. By induction, this result then generalizes to all Legendrian tangles.*

Proof. By the previous lemma, the only nontrivial case is when E is a single right cusp, when we obtain a short exact sequence of $\mathbb{Z}/m$-graded complexes $0 \to \mathrm{Span}\{e_k, d_L e_k\} \to (C(E_L), d_L) \to (C(E_R), d_R) \to 0$ with the first term acyclic. Pass to the long exact sequence of homologies, we then obtain the desired isomorphism from $H_*((C(E_L), d_L))$ to $H_*(C(E_R), d_R)$. □

Definition 3.3.4. *Given any elementary Legendrian tangle (E, μ), a m-*graded *A-*form MCS *for E is a triple $((C(E_L), d_L), \varphi, (C(E_R), d_R))$, where $(C(E_L), d_L), (C(E_R), d_R)$ are $\mathbb{Z}/m$-graded filtered complexes, $\varphi : (C(E_L), d_L) \to (C(E_R), d_R)$ is a $\mathbb{Z}/m$-graded morphism of complexes (or equivalently, the diagram φ), such that they satisfy the conditions in each case of Lemma 3.3.2. In particular, $(C(E_R), d_R)$ is determined by $(C(E_L), d_L)$ and φ.*

Remark 3.3.5. *With the definition above, Lemma 3.3.2 then shows that, there's an* identification *between the augmentation variety* $\mathrm{Aug}_m(E;k)$ *and the set of m-graded A-form MCSs* $MCS^A_m(E;k)$ *for E, for any elementary Legendrian tangle* (E,μ).

For any Legendrian tangle (T,μ), *by cutting T into elementary Legendrian tangles, one can define a m-graded A-form MCS for T as a "composition" of m-graded A-form MCSs for the elementary parts of T. We can similarly define the set* $MCS^A_m(T;k)$ *of all m-graded A-form MCSs for T. The lemma then shows by induction that, there's an identification* $\mathrm{Aug}_m(T;k) \cong MCS^A_m(T;k)$.

Handleslide moves

There're some identities involving the elementary transformations (represented by handleslides H_r or crossings s_k as in Lemma 3.3.2) between $\mathbb{Z}/m$-graded complexes. They can be represented by the *local moves* (or *handleslide moves*) of diagrams as in Figure 3.3.1: Each diagram represents a composition of elementary transformations with the maps going from left to right, and each local move represents an identity between 2 different compositions.

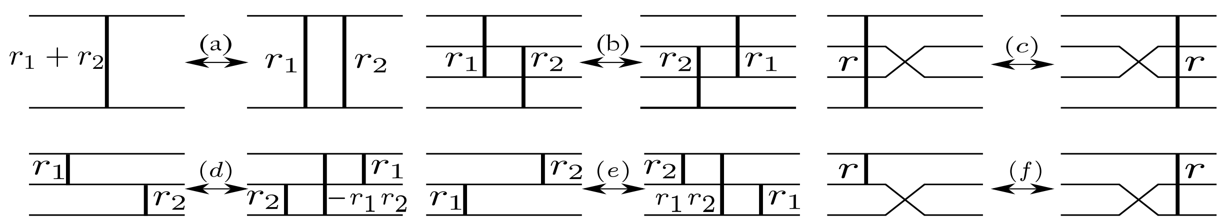


Figure 3.3.1: Local moves of handleslides in a Legendrian tangle T = identities between different compositions of elementary transformations. The moves shown do not illustrate all the possibilities.

More precisely, the possible local moves in a Legendrian tangle (T,μ) are as follows (see also [15, Section.6]):

Type 0: (Introduce or remove a trivial handleslide) Introduce or remove a handleslide with coefficient 0 and endpoints on two strands with the same Maslov potential value modulo m.

Type 1: (Slide a handleslide past a crossing) Suppose T contains one single crossing between strands k and $k+1$, and exactly one handleslide h between strands $i<j$, with $(i,j)\neq(k,k+1)$. We may slide h (either left or right) past the crossing such that the endpoints of h remain on the same strands of T. See Figure 3.3.1 (c),(f) for two such examples.

Type 2: (Interchange the positions of two handleslides) If T contains exactly two handleslides h_1, h_2 between strands $i_1<j_1$, and $i_2<j_2$, with coefficients r_1, r_2 respectively. If

$j_1 \neq i_2$ and $i_1 \neq j_2$, we may interchange the positions of the handleslides, see Figure 3.3.1 (b) for an example; If $j_1 = i_2$ (resp. $i_1 = j_2$) and h_1 is to the left of h_2, we may interchange the positions of h_1, h_2, and introduce a new handleslide between strands i_1, j_2 (resp. i_2, j_1), with coefficient $-r_1r_2$ (resp. r_1r_2), see Figure (d) (resp. (e)).

Type 3: (Merge two handleslides) Suppose T contains exactly two handleslides h_1, h_2 between the same two strands, with coefficients r_1, r_2, respectively. We may merge the two handleslides into one between the same two strands, with coefficient $r_1 + r_2$, see Figure 3.3.1 (a).

Type 4: (Introduce two canceling handleslides) Suppose T contains no crossings, cusps or handleslides. We may introduce two new handleslides between the same two strands, with coefficients $r, -r$, where $r \in k$.

Suppose T contains no crossings or cusps, recall that as usual the strands of T are labeled from top to bottom as $1, 2, \ldots, s$. Given a handleslide h in T, denote by $t_h < b_h$ the top and bottom strands of h.

Definition 3.3.6. *Given 2 handleslides h, h' in T, we say $h < h'$ if either $t_h > t_{h'}$ or $t_h = t_{h'}$ and $b_h < b_{h'}$.*
A collection of handleslides V in T is called properly ordered *if given any 2 handleslides h, h' in V, with h to the left of h', then $h < h'$.*
Given a collection of handleslides V in T, define V^t to be the collection obtained from reversing the ordering of the x-coordinates of the handleslides in V.

Assume V is a collection of handleslides in T such that either V or V^t is *properly ordered*. There're 2 additional types of moves involving V, as a composition of Type 0, 2 and 3 moves:

Type 5: (Incorporate a handleslide h into a collection V) Suppose h is a handleslide in T immediately to the right of V, with coefficient r. We move h into V via Type 0, 2 and 3 moves to create a new collection $\overline{V}$, so that $\overline{V}$ has the *same ordering property* as V: If necessary, use a *Type 0* move to introduce a trivial handleslide in V with endpoints on the same strands as h, such that V has the same ordering property as before. In this way, V contains a unique handleslide h' with endpoints on the same strands as h and say, with coefficient r' (r' may be 0); Label the handleslides between h and h' from right to left by $h_1, h_2, \ldots, h_n$. For each $1 \leq j \leq n$, move h past h_j via a *Type 2* move, which possibly creates a new handleslide h'_j (See Figure 3.3.1 (d) and (e)); Merge h'_j with the existing handleslides in V with the same endpoints as h'_j via *Type 2* moves and one *Type 3* move. The ordering property of V ensures this does not introduce any new handleslides; After the above moves, h and h' are next to each other, use a *Type 3* move to merge h and h'. The resulting handleslide has coefficient $r + r'$.

When h is immediately to the left of V, a similar procedure can be used to incorporate the handleslide h into V such that the resulting collection $\overline{V}$ has the same ordering property as V.

Type 6: (Remove a handleslide h from a collection V) Suppose h is a handleslide with coefficient r in V. Use *Type 2* moves, we can remove h from V with coefficient unchanged, so that it appears either to the left or right of the remaining handleslides (with possibly new handleslides, see Figure 3.3.1 (d), (e)), denoted by $\overline{V}$; Use *Type 2* and *Type 3* moves to reorder and merge handleslides in $\overline{V}$ so that $\overline{V}$ has the same ordering property as V. The ordering property of V ensures this can be done without introducing any new handleslides.

Augmentation varieties for elementary Legendrian tangles

Now, we're able to prove Lemma 3.2.3.

Proof of Lemma 3.2.3. The result is trivial if E is a left cusp, a marked right cusp or $2n$ parallel strands with a single base point. Also , if E is a single crossing q and $|q| \neq 0(\text{mod} m)$, then by definition, $\chi(\rho) = r(\rho) = B = 0$ and $\text{Aug}_m^\rho(E, \epsilon_L; k) = \{(\epsilon_L, \epsilon(q)) | \epsilon(q) = 0\}$ is a single point, the result follows. Now, assume E is a single crossing q between strands $k, k+1$ of E_L and $|q| = 0(\text{mod} m)$. Since $B = 0$ and $-\chi(\rho) = s(\rho) - c_R = s(\rho)$, we need to show: $\text{Aug}_m^\rho(E, \epsilon_L; k) \cong (k^*)^{s(\rho)} \times k^{r(\rho)}$.

Let $(C(E_L), d_L)$ be the complex corresponding to ϵ_L, under the identification in Remark 3.3.5, we have

$$\text{Aug}_m^\rho(E, \epsilon_L; k) = \{r \in k | (C(E_R), d_R) := s_k \circ H_r((C(E_L), d_L)) \in \mathcal{O}_m(\rho_R; k)\} \qquad (3.3.0.1)$$

Here $r = -\epsilon(q)$ corresponds to $\epsilon \in \text{Aug}_m^\rho(E, \epsilon_L; k)$ and H_r is the handleslide (which represents an elementary transformation) with coefficient r between strands $k, k+1$ of E_L. Use the identification (3.3.0.1) above, given any r in $\text{Aug}_m^\rho(E, \epsilon_L; k)$, denote $(C(E_R), d_R) := s_k \circ H_r((C(E_L), d_L))$. For simplicity, we simply write $d_R = (s_k \circ H_r) \cdot d_L$.

Firstly, we show the lemma in the case *when ϵ_L is a standard augmentation*, or equivalently the complex $(C(E_L), d_L)$ is standard (See Remark 3.1.8 for the definition).

Notice that $\rho_L(k) \neq k+1$. Let $A = \{k, k+1, \rho_L(k), \rho_L(k+1)\}$, $\alpha = \min A, \beta = \min(A \setminus \{\alpha, \rho_L(\alpha)\})$. Let $a =< d_L e_\alpha, e_{\rho_L(\alpha)} >, b =< d_L e_\beta, e_{\rho_L(\beta)} >$. Notice that both a and b are nonzero, as $(C(E_L), d_L)$ is standard with respect to ρ_L (see Remark 3.1.8). Given any r in $\text{Aug}_m^\rho(E, \epsilon_L; k)$, we divide the discussion into several cases:

(1) $\rho_L(k) < k < k+1 < \rho_L(k+1)$. If $r = 0$, then d_R is standard and q is a m-graded departure of ρ.
If $r \neq 0$, then $H \cdot d_R$ is standard for some composition of m-graded handleslides H (See Figure 3.3.2 (S1)). Notice that any m-graded handleslide represents a m-graded filtration preserving automorphism of a m-graded filtered complex, which doesn't change the isomorphism class, hence the m-graded normal ruling determined by the complex. It follows that $H \cdot d_R$ and d_R determines the same m-graded normal ruling of E_R. Hence, q is a (S1) switch of ρ (See Figure 3.3.2 (S1)).

(2) (a) $\rho_L(k+1) < \rho_L(k) < k < k+1$ or

(b) $k < k+1 < \rho_L(k+1) < \rho_L(k)$.

If $r = 0$, then d_R is standard (with respect to ρ_R), and q is a *m-graded departure* of ρ. If $r \neq 0$, then $H \cdot d_R$ is standard for $H =$ some composition of handleslides (See Figure 3.3.2 (S2) (resp. (S3))). And it follows that q is a *(S2) (resp. (S3)) switch* in the case (2a) (resp. (2b)).

(3) $\rho_0(k+1) < k < k+1 < \rho_0(k)$. Then d_R is standard and q is a m-graded *(R1) return* (See Figure 3.3.2 (R1)).

(4) (a) $\rho_0(k) < k < k+1 < \rho_0(k+1)$ or

(b) $k < k+1 < \rho_0(k) < \rho_0(k+1)$.

Then d_R is standard and q is a m-graded *(R2) (resp. (R3)) return* in the case (4a) (resp. (4b)) (See Figure 3.3.2 (R2) (resp. (R3))).

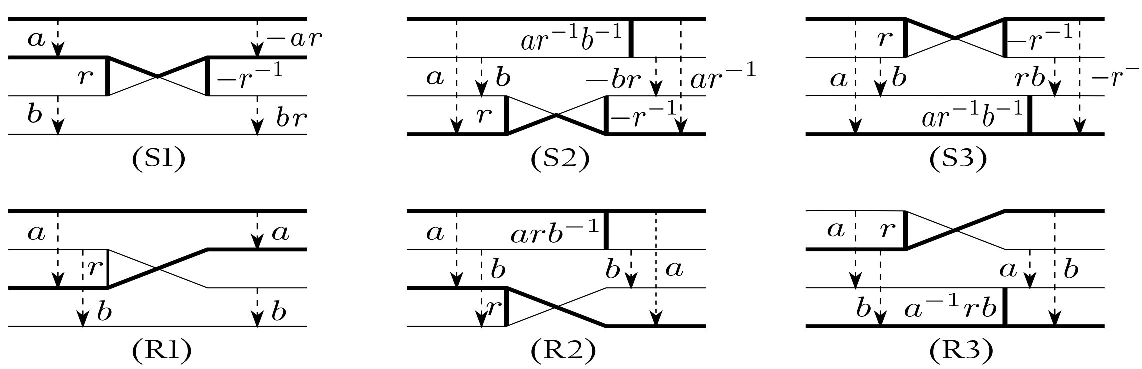


Figure 3.3.2: The handleslide moves which preserve standard complexes: the crossing is a switch on the top row and a m-graded return on the bottom row respectively, of the normal ruling determined by the complexes on the 2 ends. The dashed arrows correspond to the nonzero coefficients $< de_i, e_j >$ of the complexes associated to the 2 ends.

As a consequence, via the identification (3.3.0.1) we obtain that

$$\mathrm{Aug}_m^{\rho}(E, \epsilon_L; k) = \begin{cases} \{r | r = 0\} & \text{If } q \text{ is a } m\text{-graded departure of } \rho; \\ \{r | r \neq 0\} & \text{If } q \text{ is a switch of } \rho; \\ \{r | r \in k\} & \text{If } q \text{ is a } m\text{-graded return of } \rho. \end{cases}$$

If follows that $\mathrm{Aug}_m^{\rho}(E, \epsilon_L; k) \cong (k^*)^{s(\rho)} \times k^{r(\rho)}$, as desired.

In the general case, by Remark 3.1.8, there exists an unipotent automorphism φ_0 of $C(E_L)$ such that $d_0 := \varphi_0^{-1} \cdot d_L$ is standard. We can represent φ_0 by a collection of handleslides V which is properly ordered. Hence, $d_R = (s_k \circ H_r \circ V) \cdot d_0$. Pictorially, the morphism $s_k \circ H_r \circ V$

is represented by the Legendrian tangle front E, with a handleslide H_r to the left of q and a collection of handleslides V to the left of H_r. Let h' be the handleslide between strands $k, k+1$ in V, with coefficient r', where $r' = r'(\varphi_0)$ is a constant depending on φ_0.

Use a *type 5* move to incorporate H_r into V, to obtain a collection of handleslides V_1. The handleslide h_1 between strands $k, k+1$ in V_1 has coefficient $r + r'$. Use a *Type 6* move to remove h_1 from V_1 so that it appears to the left of the remaining handleslides, denoted by V_2. Now, V_2 contains no handleslides between strands $k, k+1$. Hence, we can use *Type 2* moves to slide V_2 past the crossing q to obtain a collection of handleslides V_3 to the right of q. The meaning of the procedure is that $s_k \circ H_r \circ V = V_3 \circ s_k \circ h_1$ as a $\mathbb{Z}/m$-graded isomorphism from $C(E_L)$ to $C(E_R)$. Hence, $d_R = (s_k \circ H_r \circ V) \cdot d_0 = (V_3 \circ s_k \circ h_1) \cdot d_0$, which is isomorphic to $(s_k \circ h_1) \cdot d_0$, where $h_1 = H_{r+r'}$ is the handleslide between strands $k, k+1$ of E_L. It follows that

$$\mathrm{Aug}_m^{\rho}(E, \epsilon_L; k) = \{r \in k | (s_k \circ H_{r+r'}) \cdot d_0 \in \mathcal{O}_m(\rho_R; k)\}.$$

Now, d_0 is standard, we have reduced the problem to the previous case. More precisely, we have

$$\mathrm{Aug}_m^{\rho}(E, \epsilon_L; k) = \begin{cases} \{r | r + r' = 0\} & \text{If } q \text{ is a } m\text{-graded departure of } \rho; \\ \{r | r + r' \neq 0\} & \text{If } q \text{ is a switch of } \rho; \\ \{r | r + r' \in k\} & \text{If } q \text{ is a } m\text{-graded return of } \rho. \end{cases} \tag{3.3.0.2}$$

and the desired result follows.

□

In Remark 3.1.8, it turns out that there's canonical choice of the unipotent automorphism φ_0:

Lemma 3.3.7. *Let (T, μ) be a trivial Legendrian tangle of n parallel strands, and ρ be any m-graded isomorphism type (Definition 3.1.6) of (T, μ). Equivalently, given the isomorphism class $\mathcal{O}_m(\rho; k) = \mathrm{Aut}(C) \cdot d_\rho$ of $\mathbb{Z}/m$-graded filtered complexes $(C = C(T), d)$ over k, determined by the Barannikov normal form d_ρ. There's a canonical algebraic map $\varphi_0 : \mathcal{O}_m(\rho; k) \to \mathrm{Aut}(C)$ such that, $\varphi_0(d)$ is unipotent and $\varphi_0(d)^{-1} \cdot d$ is standard for all $d \in \mathcal{O}_m(\rho; k)$. Equivalently, the principal bundle $\mathrm{Aut}(C) \to \mathcal{O}_m(\rho; k) = \mathrm{Aut(C)} \cdot d_\rho$ has a canonical section φ.*

Proof. As in the proof of Lemma 3.1.5, let $I = I(T) = \{1, 2, \ldots, n\}$ be the set of left endpoints of T. Then ρ determines a partition $I = U \sqcup L \sqcup H$ and a bijection $\rho : U \xrightarrow{\sim} L$, where U,L and H are the sets of *upper*,*lower* and *homological* indices in I determined by the isomorphism class $\mathcal{O}_m(\rho; k)$.

As in Definition 3.2.11, define the subsets $I(i), i \in I$ and $A(i), i \in U \sqcup H$ of I. Clearly, for each $j \in A(i)$ have $A(j) \subsetneq A(i)$.

Claim: Given any d in $\mathcal{O}_m(\rho; k)$ and any upper or homological index i in $U \sqcup H$, there exists a unique i-admissible element in C of the form $e'_i = e_i + \sum_{j \in A(i)} a_j e_j$, such that de'_i is

$\rho(i)$-admissible. When $\rho(i)=\infty$, de'_i is ∞-admissible means $de'_i=0$. Moreover, $e'_i=e'_i(d)$ depends on d algebraically.

Proof of Claim: We firstly show the *existence*. By the proof of Lemma 3.1.5, we know: $\rho(i)=\max\{\rho_d(x)|x \text{ is } i\text{-admissible.}\}$ for any $i\in U\sqcup H$. Hence, we can take an i-admissible element of the form $x=e_i+\sum_{j\in I(i)}a_je_j$, such that $\rho_d(x)=\rho(i)$, i.e. dx is $\rho(i)$-admissible.

If $a_j=0$ for all $j\in I(i)\setminus A(i)$, then $e'_i=x$ is the desired element. Otherwise, for $x_0=x$ above, *define* $j=j(x_0):=\min\{j|j\in I(i)\setminus A(i) \text{ and } a_j=<x_0,e_j>\neq 0.\}$. Then by definition of $A(i)$, either $j\in L\sqcup H$ or $j\in U$ and $\rho(j)>\rho(i)$. If $j\in L\sqcup H$, then there exists an j-admissible element of the form $y_j=e_j+\sum_{l>j}*_le_l$ such that $dy_j=0$. It follows that $x_1=x_0-a_jy_j$ is i-admissible and dx_1 is still $\rho(i)$-admissible, but $j(x_1)>j=j(x_0)$. Here, we define $\min\emptyset:=\infty$. If $j\in U$ and $\rho(j)>\rho(i)$, then there exists a j-admissible element of the form $y_j=e_j+\sum_{l>j}*_le_l$ such that dy_j is $\rho(j)$-admissible. It follows again that $x_1=x_0-a_jy_j$ is i-admissible and dx_1 is still $\rho(i)$-admissible, but $j(x_1)>j(x_0)$.

If $j(x_1)=\infty$, then $e'_i=x_1$ is the desired element in the Claim. Otherwise, replace x_0 by x_1 and repeat the procedure above. Inductively, for some sufficiently large N, we obtain in the end an i-admissible element of the form $x_N=e_i+\sum_{j\in I(i)}a_je_j$ such that dx_N is $\rho(i)$-admissible and $j(x_N)=\infty$. Now, $e'_i=x_N$ fulfils the Claim.
uniqueness. We show the uniqueness by induction on $|A(i)|$. If $A(i)=\emptyset$, then $e'_i=e_i$, which is clearly unique. For the inductive procedure, assume the uniqueness holds when $|A(i)|<k$, and consider the case when $|A(i)|=k$. Let $e'_i=e_i+\sum_{j\in A(i)}a_je_j$ be any element satisfying the Claim. Since $A(j)\subsetneq A(i)$ for all $j\in A(i)$, by induction we can rewrite $e'_i=e_i+\sum_{j\in A(i)}b_je'_j$, where e'_j is uniquely determined by d for all $j\in A(i)$. We want to show the uniqueness of b_j's.

Assume $A(i)=\{i_1<i_2<\ldots<i_k\}$ and $\rho(A(i))=\{j_1<j_2<\ldots<j_k\}\subset L$. By definition of $A(i)$, we know $\rho(i_l)<\rho(i)$ for all $1\leq l\leq k$. By the conditions of the Claim, de'_i is $\rho(i)$-admissible, hence $<de'_i,e_{\rho(i_l)}>=0$ for all $1\leq l\leq k$. That is, the following system of linear equations for b_j's holds:

$$(<e_{\rho(i_p)},de'_{i_q}>)_{p,q}(b_q)_q=(-<e_{\rho(i_p)},de_i>)_p$$

Notice that the coefficient matrix $(<e_{\rho(i_p)},de'_{i_q}>)$ is similar to $(<e_{j_p},de'_{\rho^{-1}(j_q)}>)$. And by definition, $de'_{\rho^{-1}(j_q)}$ is j_q-admissible, hence $<e_{j_p},de'_{\rho^{-1}(j_q)}>=0$ if $p<q$ and $<e_{j_q},de'_{\rho^{-1}(j_q)}>\neq 0$. That is, the square matrix $(<e_{j_p},de'_{\rho^{-1}(j_q)}>)$ is lower triangular and invertible, hence $(<e_{\rho(i_p)},de'_{i_q}>)$ is also invertible. It follows that

$$(b_q)_q=(<e_{\rho(i_p)},de'_{i_q}>)^{-1}_{p,q}(-<e_{\rho(i_p)},de_i>)_p$$

where by induction e'_{i_q}'s are uniquely determined by d, hence so is the right hand side. The uniqueness in the Claim then follows. The previous equation also shows by induction that $e'_i=e'_i(d)$ depends algebraically on d. □

On the other hand, for each $j \in L$, so $j = \rho(i)$ for some $i \in U$. By the Claim, $de'_i = c_j e'_j$ for some $c_j \neq 0$ and some j-admissible element of the form $e'_j = e_j + \sum_{l>j} *_l e_l$. The claim shows that both c_j and e'_j are also uniquely determined and depend on d algebraically. Now, define an unipotent isomorphism $\varphi_0(d)$ of C by $\varphi_0(d)(e_i) = e'_i$ for all $1 \leq i \leq n$. It follows that $\varphi_0 : \mathcal{O}_m(\rho; k) \to \mathrm{Aut}(C)$ defines a canonical algebraic map. Moreover, for any $d \in \mathcal{O}_m(\rho; k)$ and any $i \in U$, have

$$\begin{aligned} (\varphi_0(d)^{-1} \cdot d)e_i &= \varphi_0(d)^{-1} \circ d \circ \varphi_0(d)(e_i) = \varphi_0(d)^{-1} \circ d(e'_i) \\ &= c_{\rho(i)} \varphi_0(d)^{-1}(e'_{\rho(i)}) = c_{\rho(i)} e_{\rho(i)}. \end{aligned}$$

Similarly, $(\varphi_0(d)^{-1} \cdot d)e_i = 0$ for $i \in H \sqcup L$. So $\varphi_0(d)^{-1} \cdot d$ is standard.

Finally, for the canonical section of $\mathrm{Aut}(C) \to \mathcal{O}_m(\rho; k)$, we simply take $\varphi(d) := D(d) \circ \varphi_0(d)$, where $D(d)(e'_i) = e'_i$ for $i \in U \sqcup H$ and $D(d)(e'_j) = c_j e'_j$ for $j \in L$. □

Corollary 3.3.8. *Let (T, μ) be a trivial Legendrian tangle of n parallel strands, and ρ be any m-graded isomorphism type of (T, μ). Then*

$$\mathcal{O}_m(\rho; k) \cong (k^*)^{|L|} \times k^{A(\rho)}$$

where $A(\rho)$ is defined as in Definition 3.2.11.

Proof. By the previous lemma, there's an identification between $d \in \mathcal{O}_m(\rho; k)$ and $\varphi(d)$, where φ is the canonical section of $\mathrm{Aut}(C(T)) \to \mathcal{O}_m(\rho; k) = \mathrm{Aut}(C(T)) \cdot d_\rho$. But, use the notations in the proof of Lemma 3.3.7 (see also Definition 3.2.11), the general form of $\varphi(d)$ is $\varphi(d) = D(d) \circ \varphi_0(d)$, where $\varphi_0(d)(e_i) = e'_i = e_i + \sum_{j \in A(i)} *_{ij} e_j$ for $i \in U \sqcup H$ and $\varphi_0(e_i) = e_i + \sum_{j \in I(i)} *_{ij} e_j$ for $i \in L$. Moreover, $D(d)(e'_i) = e'_i$ for $i \in U \sqcup H$, and $D(d)(e'_i) = c_i e'_i$ for $i \in L$ and some $c_i \in k^*$. It follows that

$$\begin{aligned} \mathcal{O}_m(\rho; k) &\cong \{(*_{ij}, i \in U \sqcup H, j \in A(i), \text{ or } i \in L, j \in I(i), c_i, i \in L) | c_i \neq 0\} \\ &\cong (k^*)^{|L|} \times k^{A(\rho)} \end{aligned}$$

by Definition 3.2.11. □

The previous lemma allows us to show a stronger result than Lemma 3.2.3:

Lemma 3.3.9. *Let (E, μ) be an elementary Legendrian tangle: a single crossing q, a left cusp q, a marked right cusp q or $2n$ parallel strands with a single base point $*$. Let ρ be a m-graded normal ruling of E, denote $\rho_L := \rho|_{E_L}, \rho_R := \rho|_{E_R}$*[1]*. Then the natural map $P : \mathrm{Aug}_m(E, \rho_L, \rho_R; k) \to \mathcal{O}_m(\rho_L; k)$ given by $\epsilon \to \epsilon_L = \epsilon|_{E_L}$ is algebraically a* trivial fiber bundle *with fibers isomorphic to $(k^*)^{-\chi(\rho)+B} \times k^{r(\rho)}$, where B is the number of base points, $\chi(\rho) = c_R - s(\rho)$ (see Definition 2.1.4).*

[1]Notice that ρ is uniquely determined by ρ_L, ρ_R.

Proof. As in the proof of Lemma 3.2.3, the only nontrivial case is when E contains a single crossing q and $|q| = 0(\mathrm{mod} m)$. In the proof of Lemma 3.2.3 for the general case (i.e. for a general augmentation $\epsilon_L \in \mathcal{O}_m(\rho_L; k)$ or the corresponding differential d_L for $C(E_L)$), the unipotent automorphism φ_0 can be taken to be canonical: $\varphi_0 := \varphi_0(\epsilon_L)$, by lemma 3.3.7 above. Hence, the constant $r' = r'(\epsilon_l)$ in Equation (3.3.0.2) depends algebraically on ϵ_L. Use the identification in Remark 3.3.5, it follows that

$$\mathrm{Aug}_m(E, \rho_L, \rho_R) = \begin{cases} \{(\epsilon_L, r) | \epsilon_L \in \mathcal{O}_m(\rho_L), r + r'(\epsilon_L) = 0\} & q \text{ is a departure;} \\ \{(\epsilon_L, r) | \epsilon_L \in \mathcal{O}_m(\rho_L), r + r'(\epsilon_L) \neq 0\} & q \text{ is a switch;} \\ \{(\epsilon_L, r) | \epsilon_L \in \mathcal{O}_m(\rho_L)\} & q \text{ is a return.} \end{cases}$$

with the natural map P given by $(\epsilon_L, r) \to \epsilon_L$. Here we have ignored the coefficient field k. Therefore, we obtain an isomorphism

$$P \times R : \mathrm{Aug}_m(E, \rho_L, \rho_R; k) \xrightarrow{\sim} \mathcal{O}_m(\rho_L; k) \times ((k^*)^{-\chi(\rho)+B} \times k^{r(\rho)})$$

where $R(\epsilon_L, r) = r + r'(\epsilon_L)$. The result then follows. □

As a consequence, we obtain

Theorem 3.3.10. *Let (T, μ) be any Legendrian tangle, with B base points placed on T so that* each right cusp is marked. *Fix m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively. Fix $\epsilon_L \in \mathcal{O}_m(\rho_L; k)$. Then there's a decomposition of augmentation varieties into disjoint union of subvarieties*

$$\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k) = \sqcup_\rho \mathrm{Aug}_m^\rho(T, \epsilon_L, \rho_R; k)$$

(See Definition 3.2.1), where ρ runs over all m-graded normal rulings of T such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$. Moreover,

$$\mathrm{Aug}_m^\rho(T, \epsilon_L, \rho_R; k) \cong (k^*)^{-\chi(\rho)+B} \times k^{r(\rho)}. \tag{3.3.0.3}$$

Proof. Use the notations in Definition 3.2.1, $T = E_1 \circ \ldots \circ E_n$ is a composition of n elementary tangles, and the canonical projection $P_n : \mathrm{Aug}_m^\rho(T_n, \epsilon_L; k) \to \mathrm{Aug}_m^{\rho|_{T_{n-1}}}(T_{n-1}, \epsilon_L; k)$ is a base change of the projection $\mathrm{Aug}_m(E_n, \rho_{n-1}, \rho_n; k) \to \mathcal{O}_m(\rho_{n-1}; k)$. The latter, by the previous lemma, is a trivial fiber bundle with fibers isomorphic to $(k^*)^{-\chi(\rho|_{E_n})+B(E_N)} \times k^{r(\rho|_{E_n})}$. Hence, so is the projection P_n and

$$\mathrm{Aug}_m^\rho(T_n, \epsilon_L; k) \cong \mathrm{Aug}_m^{\rho|_{T_{n-1}}}(T_{n-1}, \epsilon_L; k) \times (k^*)^{-\chi(\rho|_{E_n})+B(E_n)} \times k^{r(\rho|_{E_n})}$$

By induction, the desired result then follows from Lemma 3.2.3. □

Remark 3.3.11. *We've defined varieties $\mathrm{Aug}_m^\rho(T, \rho_L, \rho_R; k) = \mathrm{Aug}_m^\rho(T, \rho_L; k)$ in Definition 3.2.1. Use Lemma 3.3.9, a similar argument as in the proof above also shows that*

$$\mathrm{Aug}_m(T, \rho_L, \rho_R; k) = \sqcup_\rho \mathrm{Aug}_m^\rho(T, \rho_L, \rho_R; k) \tag{3.3.0.4}$$

with

$$\begin{aligned} \mathrm{Aug}_m^{\rho}(T,\rho_L,\rho_R;k) &\cong \mathcal{O}_m(\rho_L;k)\times(k^*)^{-\chi(\rho)+B}\times k^{r(\rho)} \\ &\cong (k^*)^{-\chi(\rho)+B+n'_L}\times k^{r(\rho)+A(\rho_L)} \end{aligned} \tag{3.3.0.5}$$

where $n_L = 2n'_L$ *is the number of left endpoints of* T.

Remark 3.3.12. *By the previous remark, the augmentation variety* $\mathrm{Aug}_m(T,\rho_L,\rho_R;k)$ *clearly has polynomial-count [14]. By N.Katz's theorem [14, Thm.6.1.2], the counting polynomial for the points-counting of the variety* $\mathrm{Aug}_m(T,\rho_L,\rho_R;\mathbb{F}_q)$ *over finite fields, then coincides with the weight (or E-, or virtual Poincaré) polynomial of the variety* $\mathrm{Aug}_m(T,\rho_L,\rho_R;\mathbb{C})$ *over* $\mathbb{C}$*, which is again essentially computed by the ruling polynomial* $<\rho_L|R_T^m(z)|\rho_R>$*. A similar result holds for* $\mathrm{Aug}_m(T,\epsilon_L,\rho_R;k)$*.*

This gives some motivation for studying the mixed Hodge structure of the variety in the next chapter.

Chapter 4

Towards cohomology of augmentation varieties

Let (T, μ) be an oriented Legendrian tangle. The invariance of LCH DGAs up to homotopy equivalence, ensures that, given any augmentation variety with fixed boundary conditions associated to (T, μ), the mixed Hodge structure on its compactly supported cohomology, up to a normalization, is a Legendrian isotopy invariant. In this chapter, generalizing the points-counting of augmentation varieties in chapter 3, we study some more aspects of the Hodge theory of the augmentation varieties.

4.1 A spectral sequence converging to the mixed Hodge structure

In this section, associated to the ruling decomposition of the augmentation variety, we derive a spectral sequence converging to the mixed Hodge structure. As an application, we obtain some knowledge on the cohomology of the augmentation variety.

As in Section 3.2, let (T, μ) be an oriented Legendrian tangle with each right cusp marked, and $T = E_1 \circ E_2 \circ \ldots E_n$ is the composition of n elementary Legendrian tangles. Fix m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively. Denote by $\mathrm{NR}_T^m(\rho_L, \rho_R)$ the set of m-graded normal rulings ρ of T such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$.

For each $1 \leq i \leq n-1$, recall that the co-restriction of LCH DGAs induce a restriction map of augmentation varieties $r_i : \mathrm{Aug}_m(T, \rho_L, \rho_R; k) \to \mathrm{Aug}_m^a((E_i)_R = (E_{i+1})_L; k)$, where $\mathrm{Aug}_m^a((E_i)_R = (E_{i+1})_L; k)$ is the variety of acyclic augmentations (See Remark 3.1.7) of $(E_i)_R = (E_{i+1})_L$. Take the underlying normal rulings, r_i induces the restriction map on the sets of normal rulings $r_i : \mathrm{NR}_T^m(\rho_L, \rho_R; k) \to \mathrm{NR}_{(E_i)_R}^m$, given by $r_i(\rho) = \rho|_{(E_i)_R}$. Moreover, the ruling decomposition $\mathrm{Aug}_m^a((E_i)_R; k) = \sqcup_\tau \mathrm{Aug}_m^\tau((E_i)_R; k) = \mathcal{O}_m(\tau; k)$ is a stratification stratified by the $B_m((E_i)_R)$-orbits, where τ runs over the set $\mathrm{NR}_{(E_i)_R}^m$ of all m-graded normal rulings of $(E_i)_R$.

Definition 4.1.1. *Firstly, define a* geometric partial order $\leq^G$ *on* $\mathrm{NR}^m_{(E_i)_R}$ *via inclusions of strata: For any* τ, τ' *in* $\mathrm{NR}^m_{(E_i)_R}$*, we say* $\tau' \leq^G \tau$*, if* $\mathcal{O}_m(\tau';k) \subset \overline{\mathcal{O}_m(\tau;k)}$ *in* $\mathrm{Aug}^a_m((E_i)_R;k)$*. Now, define an* algebraic partial order $\leq^A$ *on* $\mathrm{NR}^m_T(\rho_L,\rho_R)$*: For any* ρ, ρ' *in* $\mathrm{NR}^m_T(\rho_L,\rho_R)$*, we say* $\rho' \leq^A \rho$*, if* $r_i(\rho') \leq^G r_i(\rho)$ *for all* $1 \leq i \leq n-1$*.*

Definition 4.1.2. *For each m-graded normal ruling* ρ *of* T*, define a* closed subvariety $A_\rho(T;k)$ *of* $\mathrm{Aug}_m(T,\rho_L,\rho_R;k)$*:*

$$A_\rho(T;k) := \{\epsilon \in \mathrm{Aug}_m(T,\rho_L,\rho_R;k) | R_T(\epsilon) \leq^A \rho\}$$

Notice that $A_\rho(T;k) = \cap_{i=1}^{n-1} r_i^{-1}(\overline{\mathcal{O}_m}(r_i(\rho);k))$*, so it's indeed a closed subvariety. It's also clear that* $A_\rho(T;k) = \sqcup_{\rho' \leq^A \rho} \mathrm{Aug}^{\rho'}_m(T;k)$ *set-theoretically.*

The ruling decomposition induces a finite *ruling filtration* of $\mathrm{Aug}_m(T,\rho_L,\rho_R;k)$ by closed subvarieties:

Definition/Proposition 4.1.3. *Define a decomposition* $\mathrm{NR}^m_T(\rho_L,\rho_R) = \sqcup_{i=0}^D R_i$ *by induction: Let* $D+1$ *be the maximal length of the ascending chains in* $(\mathrm{NR}^m_T(\rho_L,\rho_R), \leq^A)$*. Let* R_D *is the subset of maximal elements in* $(\mathrm{NR}^m_T(\rho_L,\rho_R), \leq^A)$*. Suppose we've defined* $R_{i+1},\ldots,R_D$*, let* R_i *be the subset of maximal elements in* $(\mathrm{NR}^m_T(\rho_L,\rho_R) - \sqcup_{j=i+1}^D R_j, \leq^A)$*.*
Now, define *the closed subvariety* $A_i = A_i(T,\rho_L,\rho_R;k)$ *of* $\mathrm{Aug}_m(T,\rho_L,\rho_R;k)$ *as*

$$A_i := \cup_{\rho \in R_i} A_\rho(T;k) \tag{4.1.0.1}$$

for $0 \leq i \leq D$*. By definition, we obtain a finite filtration:*

$$\mathrm{Aug}_m(T,\rho_L,\rho_R;k) = A_D \supset A_{D-1} \supset \ldots \supset A_0 \supset A_{-1} = \emptyset \tag{4.1.0.2}$$

Moreover, as varieties we have

$$A_i - A_{i-1} = \sqcup_{\rho \in R_i} \mathrm{Aug}^\rho_m(T;k) \tag{4.1.0.3}$$

That is, $A_i - A_{i-1}$ *is the disjoint union of some open subvarieties* $\mathrm{Aug}^\rho_m(T;k)$*.*

Proof. It suffices to show the last identity. This is clear set-theoretically, it's enough to show each $\mathrm{Aug}^\rho_m(T;k)$ is an open subvariety of $A_i - A_{i-1}$. We only need to show that, for any $\rho \neq \rho'$ in R_i, have $\mathrm{Aug}^\rho_m(T;k) \cap \overline{\mathrm{Aug}^{\rho'}_m(T;k)} = \emptyset$. Otherwise, say, $\epsilon \in \mathrm{Aug}^\rho_m(T;k) \cap \overline{\mathrm{Aug}^{\rho'}_m(T;k)}$, then $R_T(\epsilon) = \rho$, and $\epsilon \in \overline{\mathrm{Aug}^{\rho'}_m(T;k)} \subset r_i^{-1}(\overline{\mathcal{O}_m}(r_i(\rho');k))$ for all $1 \leq i \leq n-1$. It follows that $r_i(\epsilon) \in \overline{\mathcal{O}_m}(r_i(\rho');k)$, hence $r_i(\rho) \leq^G r_i(\rho')$ for all $1 \leq i \leq n-1$, that is, $\rho' \leq^A \rho$. However, ρ is maximal in $\mathrm{NR}^m_T(\rho_L,\rho_R) - \sqcup_{j=i+1}^D R_j$, so $\rho = \rho'$, contradiction. □

Now, the ruling filtration induces a spectral sequence computing the mixed Hodge structure (Definition/Proposition 4.2.1) of the augmentation variety $A_D = \mathrm{Aug}_m(T,\rho_L,\rho_R;\mathbb{C})$:

Lemma 4.1.4. *Any finite filtration* $A_D \supset A_{D-1} \supset \ldots \supset A_0 \supset A_{-1} = \emptyset$ *by closed subvarieties induces a* spectral sequence *converging to the compactly supported cohomology of the variety* A_D*, respecting the mixed Hodge structures*[1] *(MHS):*

$$E_1^{p,q} = H_c^{p+q}(A_p \setminus A_{p-1}) \Rightarrow H_c^{p+q}(A_D).$$

Proof. This is a well-known fact to experts. However, we give a complete proof here, due to a lack of good reference. For each $0 \le p \le D$, let $U_p = A_p - A_{p-1}$ and $j_p : U_p \hookrightarrow A_p$ be the open inclusion. Let $i_p : A_{p-1} \hookrightarrow A_p$ be the closed embedding. We then obtain a short exact sequence of sheaves on A_p:

$$0 \to (j_p)_! j_p^{-1} \underline{\mathbb{Q}}_{A_p} \to \underline{\mathbb{Q}}_{A_p} \to (i_p)_* i_p^{-1} \underline{\mathbb{Q}}_{A_p} \to 0 \tag{4.1.0.4}$$

where $\underline{\mathbb{Q}}_{A_p}$ is the constant sheaf on A_p. Take the hypercohomology with compact support, we obtain a long exact sequence in the abelian category of mixed Hodge structures (Definition/Proposition 4.2.1):

$$\ldots \to H_c^i(U_p) \xrightarrow{\alpha_p} H_c^i(A_p) \xrightarrow{\beta_p} H_c^i(A_{p-1}) \xrightarrow{\delta_p} H_c^{i+1}(U_p) \to \ldots \tag{4.1.0.5}$$

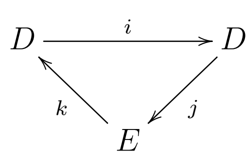
We can now construct an *exact couple* [23, Section 2.2] from the long exact sequences associated to the triples (U_p, A_p, A_{p-1}) as follows: Take

$$D := \oplus_{p,q} D^{p,q}, D^{p,q} := H_c^{p+q-1}(X_{p-1}); E := \oplus_{p,q} E^{p,q}, E^{p,q} := H_c^{p+q}(U_p).$$

Define morphisms of $\mathbb{Q}$-modules $i : D \to D$, $j : D \to E$, and $k : E \to D$ as follows: Let

$$\begin{aligned} i|_{D^{p+1,q}} &= \beta_p : D^{p+1,q} = H_c^{p+q}(X_p) \to D^{p,q+1} = H_c^{p+q}(X_{p-1}); \\ j|_{D^{p,q+1}} &= \delta_p : D^{p,q+1} = H_c^{p+q}(X_{p-1}) \to E^{p,q+1} = H_c^{p+q+1}(U_p); \\ k|_{E^{p,q+1}} &= \alpha_p : E^{p,q+1} = H_c^{p+q+1}(U_p) \to D^{p+1,q+1} = H_c^{p+q+1}(X_p). \end{aligned}$$

It's easy to check that we have obtained an exact couple $\mathcal{C} = \{D, E, i, j, k\}$ of bi-graded $\mathbb{Q}$-modules



such that the bi-degrees of the morphisms are: $\deg(i) = (-1, 1), \deg(j) = (0, 0)$, and $\deg(k) = (1, 0)$. Recall that, an *exact couple* $\mathcal{C} = \{D, E, i, j, k\}$ is a diagram of bi-graded

[1] For simplicity, we will only consider mixed Hodge structures over $\mathbb{Q}$, and the cohomology is understood as that with rational coefficients.

$\mathbb{Q}$-modules as above, with i, j, k $\mathbb{Q}$-module homomorphisms, such that, the diagram is exact at each vertex. Also, given any exact couple $\mathcal{C} = \{D, E, i, j, k\}$, the *derived couple* $\mathcal{C}' = \mathcal{C}^{(1)} = \{D', E', i', j', k'\}$ of $\mathcal{C}$ is defined as follows: Take

$$D' = i(D) = \ker(j), E' = H(E, d) = \mathrm{Ker}(j \circ k)/\mathrm{Im}(j \circ k), \text{where } d = j \circ k.$$

and define

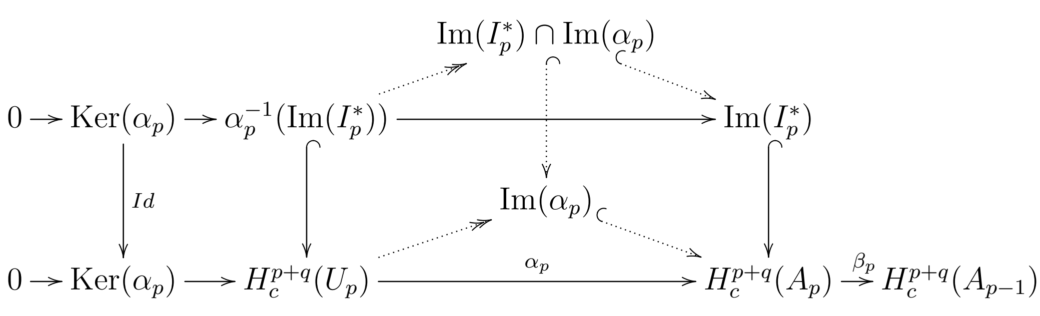
$$i' = i|_{i(D)} : D' \to D';$$
$$j' : D' \to E' \text{ by } j'(i(x)) = j(x) + dE \in E', \forall x \in D;$$
$$k' : E' \to D' \text{ by } k'(e + dE) = k(e), \forall e \in \mathrm{Ker}(d).$$

Notice that $\mathcal{C}'$ is again an exact couple [23, Prop.2.7].

In our case, for each $n \geq 0$, let $\mathcal{C}^{(n)} = \{D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)}\} = (\mathcal{C}^{(n-1)})'$ be the n-th derived couple of $\mathcal{C}$. Then, by [23, Thm.2.8], the exact couple $\mathcal{C}$ induces a spectral sequence $\{E_r, d_r\}, r = 1, 2, \ldots$, where $E_r = E^{(r-1)}$, and $d_r = j^{(r)} \circ k^{(r)}$ has bi-degree $(r, 1-r)$. In particular, $E_1 = E = E^{*,*}, d_1 = j \circ k$.

To finish the proof of the lemma, we also need to determine $E_\infty = E_r$ for $r >> 0$. By [23, Prop.2.9], have $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$, where $Z_r^{p,q} = k^{-1}(\mathrm{Im}(i^{r-1}) : D^{p+r,q-r+1} \to D^{p+1,q})$, $B_r^{p,q} = j(\mathrm{Ker}(i^{r-1}) : D^{p,q} \to D^{p-r+1,q+r-1})$. Moreover, $E_\infty^{p,q} = \cap_r Z_r^{p,q}/\cup_r B_r^{p,q}$.

In our case, clearly have $E_\infty^{p,q} = E_r^{p,q}$ for $r >> 0$. Moreover, for $r >> 0$, we see that $i^{r-1} = 0 : D^{p,q} = H_c^{p+q-1}(A_{p-1}) \to D^{p-r+1,q+r-1} = H_c^{p+q-1}(A_{p-r} = \emptyset) = 0$, and $j = \delta_p : D^{p,q} = H_c^{p+q-1}(A_{p-1}) \to E^{p,q} = H_c^{p+q}(U_p)$, so $B_r^{p,q} = \mathrm{Im}(\delta_p : H_c^{p+q-1}(A_{p-1}) \to H_c^{p+q}(U_p)) = \mathrm{Ker}(\alpha_p : H_c^{p+q}(U_p) \to H_c^{p+q}(A_p))$. On the other hand, for $r >> 0$, $i^{r-1} = I_p^* : D^{p+r,q-r+1} = H_c^{p+q}(A_{p+r-1} = A_D) \to D^{p+1,q} = H_c^{p+q}(A_p)$ is the natural morphism induced by the inclusion $I_p : A_p \hookrightarrow A_D$, and $k = \alpha_p : E^{p,q} = H_c^{p+q}(U_p) \to D^{p+1,q} = H_c^{p+q}(A_p)$. So, $Z_r^{p,q} = \alpha_p^{-1}(I_p^* : H_c^{p+q}(A_D) \to H_c^{p+q}(A_p))$. Therefore, we have $E_r^{p,q} = \alpha_p^{-1}(\mathrm{Im}(I_p^*))/\mathrm{Ker}(\alpha_p) \cong \mathrm{Im}(I_p^*) \cap \mathrm{Im}(\alpha_p) = \mathrm{Im}(I_p^*) \cap \mathrm{Ker}(\beta_p)$, where the last 2 equalities follow from the following commutative diagram with exact rows, in which all the squares are fiber products:



Let $F^pH_c^{p+q}(X_D) := \mathrm{Ker}(I_{p-1}^*)$. Clearly, the identity of inclusions $I_{p-1} = I_p \circ i_p : A_{p-1} \overset{i_p}{\hookrightarrow} A_p \overset{I_p}{\hookrightarrow} A_D$ induces $I_{p-1}^* = i_p^* \circ I_p^* = \beta_p \circ I_p^* : H_c^{p+q}(X_D) \xrightarrow{I_p^*} H_c^{p+q}(A_p) \xrightarrow{\beta_p} H_c^{p+q}(A_{p-1})$. So we obtain a filtration $H_c^{p+q}(X_D) = F^0 \supset F^1 \supset \ldots \supset F^D \supset F^{D+1} = 0$ for $H_c^{p+q}(X_D)$. Thus, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & 0 & \dashrightarrow & E_\infty^{p,q} & \cdots & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(I_p^*) & \longrightarrow & H_c^{p+q}(X_D) & \longrightarrow & \mathrm{Im}(I_p^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow Id & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(I_{p-1}^*) & \longrightarrow & H_c^{p+q}(X_D) & \longrightarrow & \mathrm{Im}(I_{p-1}^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & \dashrightarrow & F^p/F^{p+1} & \dashrightarrow & 0 & & & &
\end{array}$$

By the five lemma, we then have the natural isomorphism $E_\infty^{p,q} \cong F^p/F^{p+1}(H_c^{p+q}(X_D))$. Thus, the spectral sequence $\{E_r^{p,q}, d_r\}$ converges to $H_c^{p+q}(X_D)$, with the first page given by $E_1^{p,q} = H_c^{p+q}(U_p) = H_c^{p+q}(A_p \backslash A_{p-1})$. Finally, the compatibility with MHS is automatic, as all the morphisms in the previous construction, hence in the spectral sequence, are morphisms in the abelian category of mixed Hodge structures over $\mathbb{Q}$. □

4.2 Applications

The spectral sequence in the previous subsection allows us to draw some conclusions about the cohomology of the augmentation variety. We begin with some preliminaries on mixed Hodge structures, mainly due to Deligne [5, 6]. A general reference is [28]. We only review the part which is most relevant to us.

Definition/Proposition 4.2.1. *([5, 6] or [28])*

1. *Let X be a complex algebraic variety, for each j there exists an increasing weight filtration*

$$0 = W_{-1} \subset W_0 \subset \ldots \subset W_j = H_c^j(X) = H_c^j(X, \mathbb{Q})$$

and a decreasing Hodge filtration

$$H_c^j(X)^{\mathbb{C}} = H_c^j(X, \mathbb{C}) = F^0 \supset F^1 \supset \ldots \supset F^m \supset F^{m+1} = 0$$

such that the filtration F induces a pure Hodge structure of weight l on the complexification of the graded pieces $\mathrm{Gr}_l^W = W_l/W_{l-1}$ of the weight filtration: for each $0 \le p \le l$, we have

$$\mathrm{Gr}_l^{W^{\mathbb{C}}} = F^p \mathrm{Gr}_l^{W^{\mathbb{C}}} \oplus \overline{F^{l-p+1} \mathrm{Gr}_l^{W^{\mathbb{C}}}}.$$

2. *If X is smooth and projective, then $H_c^j(X) = H^j(X, \mathbb{Q})$ is a pure Hodge structure of weight j, with the Hodge filtration $F^i H^j(X, \mathbb{C}) = \oplus_{p+q=j, p \ge i} H^{p,q}(X)$, induced from the classical Hodge decomposition $H^j(X, \mathbb{C}) = \oplus_{p+q=j} H^{p,q}(X) = H^q(X, \Omega^p)$.*

For example, if $X = \mathbb{P}^1(\mathbb{C})$, *then* $H^2(X) = \mathbb{Q}(-1)$ *is the pure Hodge structure of weight* 2 *on* $\mathbb{Q}$, *with the Hodge filtration on* $H^2(X,\mathbb{C}) = H^{1,1}(X) = \mathbb{C}$ *given by* $F^1 = \mathbb{C}, F^2 = 0$. *Here* $\mathbb{Q}(-1)$ *is called the* (-1)-th Tate twist *(of the trivial weight 0 pure Hodge structure* $\mathbb{Q}$*). In general,* define $\mathbb{Q}(-m) := (\mathbb{Q}(-1))^{\otimes m}$ *to be the* $(-m)$-th Tate twist, *that is, a pure Hodge structure of weight* $2m$ *on* $\mathbb{Q}$, *with Hodge filtration* $F^m = \mathbb{C}, F^{m+1} = 0$.

3. *If we replace* $H^j_c(X,\mathbb{Q})$ *by any finite dimensional vector spaces* V *over* $\mathbb{Q}$, *then (1) gives a mixed* $\mathbb{Q}$*-Hodge structure (MHS) on* V. *One standard fact is that, the category of MHSs form an abelian category [28, Cor.2.5].*

4. *Given any triple* (U, X, Z) *of complex varieties, with* $i : Z \hookrightarrow X$ *the closed embedding, and* $j : U = X - Z \hookrightarrow X$ *the open complement, there exists an induced long exact sequence in the abelian category of MHSs:*

$$\ldots \to H^*_c(U) \xrightarrow{j_!} H^*_c(X) \xrightarrow{i^*} H^*_c(Z) \xrightarrow{\delta} H^{*+1}_c(U) \to \ldots$$

Definition 4.2.2. *For any complex algebraic variety* X, *define the* (compactly supported) mixed Hodge numbers *by*

$$h^{p,q;j}_c(X) := \dim_{\mathbb{C}} \mathrm{Gr}^F_p \mathrm{Gr}^W_{p+q} H^j_c(X)^{\mathbb{C}}.$$

Define the (compactly supported) mixed Hodge polynomial *of* X *by*

$$H_c(X; x, y, t) := \sum_{p,q,j} h^{p,q;j}_c(X) x^p y^q t^j.$$

And, the specialization $E(X; x, y) := H_c(X; x, y, -1)$ *is called the* weight polynomial *(or* E-polynomial*) of* X.

Definition 4.2.3. *We say, an complex algebraic variety* X *is* Hodge-Tate type, *if* $h^{p,q;j}_c = 0$ *whenever* $p \neq q$. *That is,* X *is of Hodge-Tate type, if for each* j *and* l, *the piece* $F^p \cap \overline{F^q}$ *of Hodge type* (p,q) *on the pure Hodge structure* $Gr^W_l H^j_c(X)^{\mathbb{C}}$ *vanishes whenever* $p \neq q$.

Now, we come back to the study of augmentation varieties:

Proposition 4.2.4. *The MHS on* $H^*_c(\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C}))$ *is of Hodge-Tate type.*

Proof. By the previous subsection, the ruling filtration for $A_D = \mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})$ induces a spectral sequence $E^{p+q}_1 = H^{p+q}_c(A_p \setminus A_{p-1}) \Rightarrow H^{p+q}_c(A_D)$, in the abelian category of mixed Hodge structures over $\mathbb{Q}$. Moreover, $A_p \setminus A_{p-1} = \sqcup_{\rho \in R_p} \mathrm{Aug}^{\rho}_m(T; \mathbb{C})$, where $\mathrm{Aug}^{\rho}_m(T; \mathbb{C}) = \mathrm{Aug}^{\rho}_m(T, \rho_L, \rho_R; \mathbb{C}) \cong (\mathbb{C}^\times)^{a(\rho)} \times \mathbb{C}^{b(\rho)}$ by Theorem 3.3.10, with $a(\rho) = -\chi(\rho) + B + n'_L, b(\rho) = r(\rho) + A(\rho_L)$. Hence, $E^*_1 = H^*_c(A_p \setminus A_{p-1}) = \oplus_{\rho \in R_p} H^*_c(\mathbb{C}^\times)^{\otimes a(\rho)} \otimes_{\mathbb{Q}} H^*_c(\mathbb{C})^{\otimes b(\rho)}$, is of Hodge-Tate type (Example 4.2.7). As each E^*_{r+1} is a subquotient of E^*_r, it follows that E^*_r for all $r \geq 1$, in particular, $E^*_\infty = H^*_c(A_D)$, is also of Hodge-Tate type. □

Also, we have:

Proposition 4.2.5. *$H_c^i(\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})) = 0$ for $i < C$, where $C = C(T, \rho_L, \rho_R) := (-\chi(\rho)+B+n'_L)+2(r(\rho)+A(\rho_L))$ (Remark 3.3.11) is a constant depending only on T, ρ_L, ρ_R. In particular, the cohomology $H_c^*(\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C}))$ vanishes in the lower-half degrees.*

Proof. In the proof of Proposition 4.2.4, we observe that $H_c^*((\mathbb{C}^\times)^{a(\rho)} \times \mathbb{C}^{b(\rho)}) = 0$ if $* < a(\rho) + 2b(\rho) = C$ (Example 4.2.7). Hence, $E_1^{p,q} = H_c^{p+q}(A_p \setminus A_{p-1}) = 0$ if $p + q < C$. It then follows from the spectral sequence that, E_r^{p+q} for all $r \geq 1$, in particular, $E_\infty^{p+q} = H_c^{p+q}(A_D)$, vanishes if $p + q < C$. □

Remark 4.2.6. *Notice that, by Theorem 3.3.10, if we instead work with the augmentation varieties $\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k)$, all the previous discussions in this section still apply, possibly up to a different normalization.*

Examples

Example 4.2.7. *We begin with some preliminary examples.*

1. *Take $X = \mathbb{C}^\times$. For example, take T to be the standard Legendrian unknot with 2 base points, with one on the right cusp, then $X = \mathrm{Aug}_m(T; \mathbb{C}) = \mathbb{C}^\times$. Let $Y = \mathbb{P}^1(\mathbb{C})$, and $j : X = \mathbb{C}^\times \hookrightarrow Y$ be the open inclusion, with the closed complement $i : Z = \{0, \infty\} \hookrightarrow Y$. From the classical Hodge theory, we know $H_c^*(Y) = \mathbb{Q}[0] \oplus \mathbb{Q}(-1)[-2]$, where $[\cdot]$ corresponds to the cohomological degree shifting. That is, $H_c^*(Y)$ is the pure Hodge structure $\mathbb{Q}$ in cohomology degree 0, $\mathbb{Q}(-1)$ in cohomology degree 2, and 0 otherwise. Similarly, $H_c^*(Z) = \mathbb{Q}^2[0]$. Now, by Definition/Proposition 4.2.1, the triple (X, Y, Z) induces a long exact sequence of mixed Hodge structures:*

$$\begin{aligned} 0 &\to H_c^0(X) \to H_c^0(Y) = \mathbb{Q} \to H_c^0(Z) = \mathbb{Q}^2 \\ &\to H_c^1(X) \to H_c^1(Y) = 0 \to H_c^1(Z) = 0 \\ &\to H_c^2(X) \to H_c^2(Y) = \mathbb{Q}(-1) \to H_c^2(Z) = 0 \end{aligned}$$

   Together with the knowledge about the cohomology of X, it implies that $H_c^*(X) = \mathbb{Q}[-1] \oplus \mathbb{Q}(-1)[-2]$ as MHSs. Thus, $H_c(X; x, y, t) = t + xyt^2$.

2. *Similarly, take $X = \mathbb{C}$. We see that $H_c^*(X) = \mathbb{Q}(-1)[-2]$. Thus, $H_c(X; x, y, t) = xyt^2$.*

3. *Now, take $X = (\mathbb{C}^\times)^a \times \mathbb{C}^b$. The Künneth formula implies that $H_c^*(X) = H_c^*(\mathbb{C}^\times)^{\otimes a} \otimes_{\mathbb{Q}} H_c^*(\mathbb{C})^{\otimes b} = (\mathbb{Q}[-1] \oplus \mathbb{Q}(-1)[-2])^{\otimes a} \otimes_{\mathbb{Q}} (\mathbb{Q}(-1)[-2])^{\otimes b}$. Thus, $H_c(X; x, y, t) = (t + xyt^2)^a (xyt^2)^b$. In particular, X is of Hodge-Tate type, and $H_c^*(X)$ vanishes if $* < a+2b$.*

Example 4.2.8. *Take (Λ, μ) be the Legendrian right-handed trefoil knot as in Figure 4.2.1, with $B(\Lambda) = 2$ base points placed on the 2 right cusps. Clearly, the rotation number $r = 0$. As in the figure, denote the generic vertical lines by $x = x_i, 0 \leq i \leq 3$. Take the*

Legendrian tangle $(T,\mu):=(\Lambda,\mu)|_{\{x_0<x<x_3\}}$, *this is Example 3.2.13. So,* $T=E_1\circ E_2\circ E_3$ *is a composition of 3 elementary Legendrian tangles, where* $E_i=\Lambda|_{\{x_{i-1}<x<x_i\}}$ *for* $1\leq i\leq 3$. *Denote* $T_i=\Lambda|_{\{x_0<x<x_i\}}=E_1\circ\ldots\circ E_i$ *for* $1\leq i\leq 3$. *As usual, for each* i, *label the strands of* T *over* $x=x_i$ *from top to bottom by* $1,2,\ldots,s_i$. *For simplicity, take* $m\neq 1$. *Recall [33, Example.2.13] that* $\mathrm{NR}^m_{T_L}=\{(\rho_L)_1=(12)(34),(\rho_L)_2=(13)(24)\}$, $\mathrm{NR}^m_{T_R}=\{(\rho_R)_1=(12)(34),(\rho_R)_2=(13)(24)\}$.

Use the notations in Example 3.2.13, recall that

(1). $\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_1;k)\cong\{(x_i)_{i=1}^3\in k^3|x_1+x_3+x_1x_2x_3\neq 0\}=k^*\times k\sqcup k^*\times k\sqcup(k^*)^3$.

(2). $\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_2;k)\cong\{(x_i)_{i=1}^3\in k^3|x_1+x_3+x_1x_2x_3=0\}\cong\{(x_i)_{i=1}^2\in k^2|1+x_1x_2\neq 0\}=k\sqcup(k^*)^2$.

(3). $\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_1;k)\cong\{(x_i)_{i=1}^3\in k^3|1+x_2x_3\neq 0\}=k^2\sqcup k\times(k^*)^2$.

(4). $\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_2;k)=\{(x_i)_{i=1}^3|1+x_2x_3=0\}\cong\{(x_i)_{i=1}^2\in k^2|x_2\neq 0\}=k\times k^*$.

where in each case above, the last equality corresponds to the ruling decomposition. Also, (x_1,x_2,x_3) *corresponds to the augmentation* $\epsilon\in\mathrm{Aug}_m(T;k)$ *defined by* $\epsilon(a_i)=x_i$, *and* $\epsilon|_{T_L}=\epsilon_{(\rho_L)_1}$ *(resp.* $\epsilon_{(\rho_L)_2}$*) in* $(1),(2)$ *(resp.* $(3),(4)$*).*

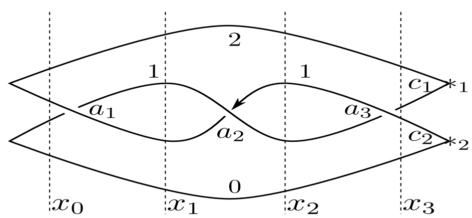


Figure 4.2.1: (Λ,μ) = the Legendrian right-handed trefoil knot with 2 base points $*_1,*_2$ at the right cusps c_1,c_2 respectively. a_1,a_2,a_3 are the crossings, and the numbers encode the Maslov potential values on each strand. Moreover, define Legendrian tangles $T_i:=\Lambda|_{\{x_0<x<x_i\}},1\leq i\leq 3$, and $T=T_3$.

We want to compute the mixed Hodge polynomial for each case. Define *for each Legendrian tangle* T_i, *the augmentation variety* $\mathrm{Aug}_m(T_i,\epsilon_L;k):=\{\epsilon\in\mathrm{Aug}_m(T_i;k):\epsilon|_{(T_i)_L}=\epsilon_L\}$. Denote $k:=\mathbb{C}$. *For each* i, denote *the pairs of endpoints of* $T|_{\{x=x_i\}}$ *by* a^i_{pq}, $1\leq p<q\leq 4$. *In the computation, we will use the following fact frequently: If* $Y=\mathbb{A}^n(k)$ *is an affine space,* $j:U\hookrightarrow Y$ *is an nonempty Zariski open subset, and* $i:Z=Y-U\hookrightarrow Y$ *is the closed complement. Then* $H^*_c(Y)=\mathbb{Q}(-n)[-2n]$ *as MHSs, and* $\dim_k Z\leq n-1\Rightarrow H^{2n-1}_c(Z)=0=H^{2n}_c(Z)$. *Thus, by Definition/Proposition 4.2.1, the triple* (U,Y,Z) *induces exact sequences of MHSs:*

$$0=H^i_c(Y)\to H^i_c(Z)\xrightarrow{\sim}H^{i+1}_c(U)\to H^{i+1}_c(Y)=0, i+1<2n; \quad (4.2.0.1)$$
$$0=H^{2n-1}_c(Z)\to H^{2n}_c(U)\xrightarrow{\sim}H^{2n}_c(Y)=\mathbb{Q}(-n)\to H^{2n}_c(Z)=0.$$

(a). Firstly, consider case (2). Note: $\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_2; k) \cong \mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}, (12)(34); k)$, *where* $(12)(34)$ *is the* m*-graded normal ruling of* $(T_2)_R$. *It's direct to see that* $\mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}; k) = \{(x_i)_{i=1}^2 \in k^2\} \cong k^2$. *Notice that* $1 + x_1x_2 = \epsilon(a_{12}^2)$ *for any* $\epsilon \in \mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}; k)$, *it follows that* $j : \mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}, (12)(34); k) = \{1 + x_1x_2 \neq 0\} \hookrightarrow \mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}; k)$ *is the open embedding, and* $i : \mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}, (13)(24); k) = \{1 + x_1x_2 = 0\} \cong k^\times \hookrightarrow \mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}; k)$ *is the closed complement. Hence, (4.2.0.1) implies that*

$$H_c^*(\mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}, (12)(34); k)) \cong H_c^{*-1}(k^\times) = (\mathbb{Q}[-1] \oplus \mathbb{Q}(-1)[-2])^{*-1}, * < 4;$$
$$H_c^4(\mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}, (12)(34); k)) \cong H_c^4(k^2) = \mathbb{Q}(-2).$$

Thus,

$$H_c^*(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_2; k)) = \mathbb{Q}[-2] \oplus \mathbb{Q}(-1)[-3] \oplus \mathbb{Q}(-2)[-4].$$

(b). Consider the case (1). Note: $\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}; k) = \{(x_i)_{i=1}^3 \in k^3\} \cong k^3$, *and* $x_1 + x_3 + x_1x_2x_3 = \epsilon(a_{12}^3)$ *for any* $\epsilon \in \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}; k)$. *So* $j : \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_1; k) = \{x_1 + x_3 + x_1x_2x_3 \neq 0\} \hookrightarrow \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}; k)$ *is the open embedding, and* $i : \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_2; k) = \{x_1 + x_3 + x_1x_2x_3 = 0\} \hookrightarrow \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}; k)$ *is the closed complement. Hence, (4.2.0.1) implies that*

$$H_c^*(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_1; k)) \cong H_c^{*-1}(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_2; k))$$
$$= (\mathbb{Q}[-2] \oplus \mathbb{Q}(-1)[-3] \oplus \mathbb{Q}(-2)[-4])^{*-1}, * < 6;$$
$$H_c^6(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_1; k)) \cong H_c^6(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}; k)) = \mathbb{Q}(-3).$$

That is,

$$H_c^*(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_1; k)) = \mathbb{Q}[-3] \oplus \mathbb{Q}(-1)[-4] \oplus \mathbb{Q}(-2)[-5] \oplus \mathbb{Q}(-3)[-6].$$

(c). Consider the case (4). As $\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_2; k) \cong k^\times \times k$, *by Example 4.2.7, we immediately have:*

$$H_c^*(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_2; k)) = \mathbb{Q}(-1)[-3] \oplus \mathbb{Q}(-2)[-4].$$

(d). Finally, consider the case (3). Note: $\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}; k) = \{(x_i)_{i=1}^3 \in k^3\} \cong k^3$, *and* $1 + x_2x_3 = \epsilon(a_{12}^3)$ *for any* $\epsilon \in \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}; k)$. *So* $j : \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_1; k) \cong \{(x_i)_{i=1}^3 \in k^3 | 1 + x_2x_3 \neq 0\} \hookrightarrow \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}; k)$ *is an open embedding,* $i : \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_2; k) = \{1 + x_2x_3 = 0\} \cong k^\times \times k \hookrightarrow \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}; k)$ *is the closed complement. Hence, (4.2.0.1) implies that*

$$H_c^*(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_1; k)) \cong H_c^{*-1}(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_2; k))$$
$$= (\mathbb{Q}(-1)[-3] \oplus \mathbb{Q}(-2)[-4])^{*-1}, * < 6;$$
$$H_c^6(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_1; k)) \cong H_c^6(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}; k)) = \mathbb{Q}(-3).$$

That is,

$$H_c^*(\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_1; k)) = \mathbb{Q}(-1)[-4] \oplus \mathbb{Q}(-2)[-5] \oplus \mathbb{Q}(-3)[-6].$$

Note*:* $\mathrm{Aug}_m(\Lambda; k) \cong \mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_1; k)$*, so we also have* $H_c^*(\mathrm{Aug}_m(\Lambda; k)) = \mathbb{Q}[-3] \oplus \mathbb{Q}(-1)[-4] \oplus \mathbb{Q}(-2)[-5] \oplus \mathbb{Q}(-3)[-6]$*. In particular, the mixed Hodge polynomial is given by* $H_c(\mathrm{Aug}_m(\Lambda; k); x, y, t) = t^3 + qt^4 + q^2t^5 + q^3t^6$*, where* $q = xy$*. Clearly,* $B(\Lambda) = 2$*, and* $\dim = \dim \mathrm{Aug}_m(\Lambda; k) = 3$*. It follows that,*

$$P_\Lambda^m(q, t) = (t + qt^2)^{-B(\Lambda)}(qt^2)^{-\dim + B(\Lambda)} H_c(\mathrm{Aug}_m(\Lambda; k); x, y, t) = \frac{1 + q^2t^2}{(1 + qt)qt}$$

gives the 2-variable invariant generalizing the ruling polynomial.

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