MATH 113, QUIZ #2

SEPTEMBER 3

1. Let $f : \mathbb{Z}_6 \to U_6$ be the function defined by $f(\overline{k}) = \left(e^{\frac{\pi}{3}i}\right)^{2k}$. Prove that f is a homomorphism but NOT an isomorphism (of binary structures).

Check HP: Let $\overline{k}, \overline{l} \in \mathbb{Z}_6$.

Then $f(\underbrace{\overline{k}+\overline{l}}_{\mathbb{Z}_6 \text{ op}}) = f(\overline{k+l}) = \left(e^{\pi i/3}\right)^{2(k+l)}$

and $\underbrace{f(\overline{k}) \cdot f(\overline{l})}_{U_6 \text{ op}} = \left(e^{\pi i/3}\right)^{2k} \cdot \left(e^{\pi i/3}\right)^{2l}$.

These are $=$ by exponent properties, so HP holds.

However f is not 1-1, since $f(\overline{0}) = f(\overline{3}) = 1$.

2. Let $\langle F, + \rangle$ denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ under ~~componentwise~~ pointwise addition. Is the map $\varphi : \langle F, + \rangle \to \langle F, + \rangle$ defined by $f(x) \mapsto x \cdot f(x)$ a homomorphism? Give a clear explanation.

Yes. Let $f(x), g(x) \in F$.

Then $\varphi(f(x)+g(x)) = x[f(x)+g(x)]$
$= x \cdot f(x) + x \cdot g(x)$.

and $\varphi(f(x)) + \varphi(g(x)) = x \cdot f(x) + x \cdot g(x)$.

These are $=$, so it's a homomorphism.

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3. Prove that commutativity of the binary operation is a structural property for binary algebraic structures. I.e. if $\varphi : \langle S, * \rangle \to \langle S', *' \rangle$ is an isomorphism and $*$ is commutative, then so is $*'$.

Suppose φ above is an isomorphism, and $*$ is comm.

I.e. $\forall$ $a, b \in S$, $a * b = b * a$.

WTS: $*'$ is comm.

Proof. Let $a', b' \in S$. Since φ is onto, $\exists$ $a, b \in S$ so that $a' = \varphi(a)$ and $b' = \varphi(b)$.

Since $*$ is commutative, we know $a * b = b * a$. Applying φ, we have $\varphi(a * b) = \varphi(b * a)$. Using the HP, we get $\varphi(a) *' \varphi(b) = \varphi(b) *' \varphi(a)$, i.e. $a' *' b' = b' *' a'$, which implies $*'$ is commutative, since a' and b' were arbitrary elements of S'.

Note: It is crucial to start with elements of S' here, then think about where they came from in S.