MATH 113 MIDTERM
Tuesday, October 13, 2015

This exam has 6 problems on 9 pages, including this cover sheet. There is also a blank page at the end you may tear off to use as scratch paper. The only thing you may have out during the exam is the exam itself and one or more writing utensils. You have 80 minutes to complete the exam.

DIRECTIONS

- Be sure to carefully read the directions for each problem.
- All work must be done on this exam. If you need more space for any problem, feel free to use the space on the very last page. Draw an arrow or write a note indicating this, so I know where to look for the rest of your work.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- Good luck; do the best you can!

Problem	Max	Score
1	12	
2	8	
3	10	
4	10	
5	20	
6	40	
Total	100	

Problem 0: Take a deep breath – you've got this!

1. This problem will deal with our development of cosets and kernels.

   (a) (6 points) Let G be a group (in multiplicative notation), and H a subgroup of G. Prove that the following is an equivalence relation. (This is precisely the relation for partitioning G into left cosets of H.)

$$x \sim y \qquad \text{iff} \qquad x^{-1}y \in H$$

   (b) (6 points) Let $\varphi : G \to B$ be a surjective group homomorphism with kernel K. If $b \in B$, and $\varphi(x) = b$, prove that $\varphi^{-1}[\{b\}] = xK$. (Hint: show the inclusion both ways. Note that part (a) implies $g \in xK$ if and only if $x^{-1}g \in K$. Also, feel free to just write $\varphi^{-1}[b]$ without the extra curly brackets.)

2. For all parts of this problem, let $G = \mathbb{Z}_{12} \times \mathbb{Z}_8$ and $H = \langle (10, 4) \rangle$. Show any computations you do, but explanations in complete sentences are not required.

   (a) (2 points) How many elements are in the factor group G/H?

   (b) (2 points) In FTFGAG form, list all possible isomorphism types for abelian groups of order $|G/H|$.

   (c) (2 points) Find the order of the element $(2, 1) + H$ in G/H.

   (d) (2 points) Based on your answer from (c) – and no additional computations – which groups in (b) are definitely NOT isomorphic to G/H and why?

3. (10 points) Prove the following statement. You may feel free to quote earlier results, as long as you state them carefully. (Obviously, don't simply write "We proved these statements in ungraded HW," and expect to get credit.)

   If $\varphi : G \to H$ is a group homomorphism between two finite groups, then $|\operatorname{im}\varphi|$ divides both $|G|$ and $|H|$.

4. (10 points) Prove the following statement. Feel free to quote linear algebra theorems about determinants.

   Let $G = GL(5, \mathbb{C})$, and let $N = \{M \in G : \det M = 2^k \text{ for some } k \in \mathbb{Z}\}$. Prove that $N \trianglelefteq G$.

5. This problem consists of several short answer questions (not related to each other). You do not need to give explanations in full sentences, but if you have done any computations, you work should show these. (Even easy computations like $lcm(6, 8)$.)

   (a) (5 points) Find a subgroup of order 56 in $\mathbb{Z}_{12} \times \mathbb{Z}_{35} \times \mathbb{Z}_{22}$.

   (b) (5 points) Find the order of the element $\sigma = (1, 7)(2, 7, 6)(3, 7, 5, 8)(1, 9, 4, 2, 3)$ in S_9.

Short answer continued. You do not need to give explanations in full sentences, but if you have done any computations, you work should show these.

(c) (5 points) Find the kernel of the homomorphism $f : \mathbb{Z}_{24} \to \mathbb{Z}_{30} \times D_3$ defined by $f(\overline{1}) = (\overline{5}, r)$.

(d) (5 points) Find a nontrivial homomorphism $\varphi : \mathbb{Z}_{12} \to D_{30}$.

6. (2 points each) This problem consists of TRUE/FALSE questions on a variety of topics (there are two pages of these). Remember to pick TRUE only if the statement is ALWAYS true. You will receive 0 points for a wrong answer, 1 point for leaving it blank, and 2 points for a correct answer. No justification is required, though you may use any blank space for scratch work.

   (a) **TRUE** **FALSE**
   The unit circle U in the complex plane is a subgroup of $\mathbb{C}^*$.

   (b) **TRUE** **FALSE**
   The nontrivial cyclic subgroups of $\langle \mathbb{R}^3, + \rangle$ are all infinite.

   (c) **TRUE** **FALSE**
   The lower triangular matrices in $GL(2, \mathbb{C})$ form a normal subgroup of $GL(2, \mathbb{C})$.

   (d) **TRUE** **FALSE**
   The group $GL(2, \mathbb{C})$ has at least one element of order k for every positive integer k, plus elements of infinite order.

   (e) **TRUE** **FALSE**
   The groups $\mathbb{Z}_{12} \times \mathbb{Z}_{22} \times \mathbb{Z}_{35}$ and $\mathbb{Z}_{10} \times \mathbb{Z}_{14} \times \mathbb{Z}_{66}$ are isomorphic.

   (f) **TRUE** **FALSE**
   The group S_7 has a subgroup of order 14.

   (g) **TRUE** **FALSE**
   There are 16 different isomorphisms $\phi : \mathbb{Z}_{10} \to U_{10}$.

   (h) **TRUE** **FALSE**
   In S_{10}, the subgroup H generated by the 3-element set $\{(1, 2)(3, 4), (1, 5, 7), (4, 5)(6, 8)\}$ is also a subgroup of A_{10}.

   (i) **TRUE** **FALSE**
   Permutations in A_n all have even order.

   (j) **TRUE** **FALSE**
   Every proper subgroup of S_4 is abelian.

(k) **TRUE FALSE**
If G is a finite group with $H \leq G$ and $|H| = \frac{1}{2}|G|$, then $H \trianglelefteq G$.

(l) **TRUE FALSE**
The group $D_4 \times S_3$ has at least one element of order 24.

(m) **TRUE FALSE**
If H and K are normal subgroups of G with $H \cong K$, then $G/H \cong G/K$.

(n) **TRUE FALSE**
The following is a group homomorphism: $\varphi : \mathbb{Z}_{18} \to \mathbb{Z}_{45}$ defined by $\varphi(\overline{k}) = \overline{3k}$.

(o) **TRUE FALSE**
If K is a normal subgroup of G, then there exists a group H and a group homomorphism $\phi : G \to H$ such that $K = \ker \phi$.

(p) **TRUE FALSE**
If G is a group, then the set $\{x \in G : x^2 = e\}$ is a subgroup of G.

(q) **TRUE FALSE**
Every subgroup of $\mathbb{Z}_{120}$ is cyclic, and every factor group of $\mathbb{Z}_{120}$ is cyclic.

(r) **TRUE FALSE**
The factor group $\mathbb{Z} \times \mathbb{Z}/\langle(3,3)\rangle$ is infinite, but it contains at least one non-identity element with finite order.

(s) **TRUE FALSE**
The factor group $\mathbb{Z}_4 \times \mathbb{Z}_8/\langle(2,4)\rangle$ is isomorphic to $\mathbb{Z}_4/\langle 2\rangle \times \mathbb{Z}_8/\langle 4\rangle$.

(t) **TRUE FALSE**
The factor group $\mathbb{Z}_4 \times \mathbb{Z}_8/(\langle 2\rangle \times \langle 4\rangle)$ is isomorphic to $\mathbb{Z}_4/\langle 2\rangle \times \mathbb{Z}_8/\langle 4\rangle$.

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