MATH 113, HOMEWORK #13

Due Thursday, December 3

Remember, consult the Homework Guidelines for general instructions. Keep in mind that for us, rings are always required to have a multiplicative identity.

GRADED HOMEWORK:

1. In this problem, we will build an algebraic extension of $\mathbb{Q}$ in a couple different ways.
   (a) Let $\mathbb{Q}$ be our base field. Build the extension $\mathbb{Q}(\sqrt{5}, \sqrt{7})$ (i.e. the smallest field containing $\mathbb{Q}$, $\sqrt{5}$, and $\sqrt{7}$) in two steps: first a simple extension to add in $\sqrt{5}$, and then another simple extension to add in $\sqrt{7}$ to the result. Give a basis for the overall extension from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{5}, \sqrt{7})$, as in the proof of Theorem 31.4.
   (b) Next, find the irreducible polynomial $irr(\sqrt{5} - \sqrt{7}, \mathbb{Q})$. Verify that your polynomial is irreducible over $\mathbb{Q}$. (It will have degree 4, so keep in mind that it won't be sufficient to check for roots – you also need to make sure it doesn't factor as a product of two quadratics). Find the obvious basis for $\mathbb{Q}(\sqrt{5} - \sqrt{7})$ over $\mathbb{Q}$.
   (c) Finally, show that the two extension fields you built in parts (a) and (b) are actually the same by using linear algebra – show that every element in the (a) basis can be written as a $\mathbb{Q}$-linear combination of elements in the (b) basis and vice versa.
2. Fraleigh Section 31, #25.
3. Fraleigh Section 31, #30

UNGRADED HOMEWORK:
Check out the review handouts on the website.