MATH 113, HOMEWORK #11

Due Thursday, November 12

Remember, consult the Homework Guidelines for general instructions. Keep in mind that for us, rings are always required to have a multiplicative identity.

GRADED HOMEWORK:

1. Note that if an equation in one or more variable has integer solutions, then the same equation will also have solutions in $\mathbb{Z}_n$ (by just taking everything mod n). The contrapositive of this statement thus gives us a technique for showing that an equation has no integer solutions – i.e. if we can find a prime n such that the equation has no solutions in $\mathbb{Z}_n$, then the equation has no integer solutions. (This technique is most practical for primes, but works for any integer $n \geq 2$.)

   For each of the following, use the technique above to show that the equation has no integer solutions.

   (a) $21x^2 - 36y = 44$
   (b) $3x^2 - 4y = 5$
   (c) $x^5 - 3y^5 = 2008$

2. Let $R = \mathbb{Q}[x]$.

   (a) Prove that the set I of all polynomials in R which have 2 as a zero forms an ideal of R. Which of our adjectives for ideals (maximal, prime, principal) apply to I? Justify your answer.
   (b) What familiar ring is R/I isomorphic to? Justify your answer.

3. Let $R = \mathbb{Q}[x, y]$, the ring of polynomials in two variables. (Be careful here – our nice $F[x]$ theorems do not apply to multivariable polynomial rings.)

   (a) Prove that $I = \{f(x, y) \in R : f(1, 3) = f(2, -5) = 0\}$ is an ideal of R.
   (b) Is I equal to the principal ideal $\langle(3x - y)(5x + 2y)\rangle$ in R? Justify your answer.
   (c) Can you determine whether I is maximal and/or prime?

UNGRADED HOMEWORK:

* Prove the divisibility rules for elevens: to determine whether a number is divisible by 11, take the alternating sum of its digits and see if that is divisible by 11 (the answer is either yes for both or no for both). For example, let's check if 11 divides 1513181912. We have $1-5+1-3+1-8+1-9+1-2 = 5-27 = 22$, so our number is divisible by 11. (Hint: you can write a positive integer with k digits as $a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_2 10^2 + a_1 10 + a_0$, where the a_i's are the digits.)

* Find all integers which simultaneously satisfy both of the following congruences.

$$15x \equiv 6 \mod 12$$
$$17x \equiv 12 \mod 25$$

Note: Starred problems from this list are classic results you will almost certainly need to use again.

Section	Problems
26	#22*, 24, 26, 27, 29
27	#1, 3, 7, 9, 14, 15, 16, 17 #19, 25*, 26*, 34*, 35*