HW #11 Solutions

#1a) Take everything mod 3:

$21x^2 - 36y = 44$ becomes

$\bar{0} = \bar{2}$ in $\mathbb{Z}_3$, which is clearly impossible. Thus, there are no solutions ~~&~~ in $\mathbb{Z}_3$ and therefore none in $\mathbb{Z}$.

#1b) Take everything mod 4:

$3x^2 - 4y = 5$ becomes

$3x^2 = 1$. Here we can easily check the four possibilities for $x \in \mathbb{Z}_4$.

$0^2 = 2^2 = 0$ and $1^2 = 3^2 = 1$,

so $3x^2$ is either $3 \cdot 0 = 0$ or $3 \cdot 1 = 3$, not 1. Thus, there are no possible x-values in $\mathbb{Z}_4$ and thus none in $\mathbb{Z}$.

#1c) Take everything mod 11. (Here the ②

motivation is as follows: Mod 11, there are only 2 values when x^{10} is evaluated – 0 and 1 – by Fermat's Little thm. Thus, there are only 3 values of x^5, i.e. 0, 1, and $-1 = 10$, since $\mathbb{Z}_{11}$ is an integral domain. I.e. $x^5 = \sqrt{x^{10}}$, and ~~...~~ $a^2 = 0$ and $a^2 = 1$ have only 1 and 2 solutions respectively, namely 0 and ± 1.)

then $x^5 - 3y^5 = 2008$ becomes

$x^5 - 3y^5 = 6$ (since $2002 \equiv 0 \mod 11$).

OR $x^5 + 8y^5 = 6$

Either way you write it, we can see that any combination of

$\{0, 1, -1\} - 3 \cdot \{0, 1, -1\}$ can never equal 6,

or $\{0, 1, -1\} + 8 \cdot \{0, 1, -1\}$ ~~...~~ just

by testing the possibilities.

Thus there are no solutions in $\mathbb{Z}_{11}$ or $\mathbb{Z}$.

③

#2a) We will run through our checklist for the ideal criterion (subset, $0 \in I$, closed under subtraction, closed under mult by elts of R).

- Clearly $I \subseteq R$, by definition.
- The 0 polynomial is in I, since it has every value as a root including 2.
- Suppose $f(x), g(x) \in I$. i.e. $f(2) = g(2) = 0$.

Then $(f-g)(x) = f(x) - g(x)$ $\forall x \in \mathbb{Q}$, so

$$(f-g)(2) = f(2) - g(2) = 0 - 0 = 0,$$

so $f-g \in I$.

- Suppose $f \in I$ and $h \in R$. Then

$(fh)(x) = f(x)\,h(x)$ $\forall x \in \mathbb{Q}$, so

$(fh)(2) = f(2) \cdot h(2) = 0 \cdot h(2) = 0$, so

$fh \in I$.

Thus I is an ideal of R.

By the Factor theorem, a polynomial has 2 as a zero iff it has $(x-2)$ as a factor, so we can see that $I = \langle x-2 \rangle$, i.e. I consists of precisely the polynomial multiples of $x-2$ in R. So I is principal.

I is also maximal and therefore prime.

Justification: either quote Thm: $\langle \text{irred} \rangle$ = maximal in $F[x]$, or

see part b, in which we show $R/I \cong \mathbb{Q}$, which is a field, so I is maximal by the Thm: R/I = field iff I = max'l.

#2b) $R/I \cong \mathbb{Q}$. There are several ways to show this. One slick way is by the 1st Isomorphism thm for rings.

take the evaluation homomorphism...

$$\phi_2 : R \longrightarrow \mathbb{Q}$$

$$f(x) \longmapsto f(2).$$

Evaluation is always a homomorphism, so we don't need to check that.

The kernel $\ker \phi_2$ is precisely I,

since $\phi_2^{-1}(\{0\}) = \{ f(x) \in R : f(2) = 0 \} = I$.

Finally the map is surjective, since if f is any constant polynomial $f(x) = c$ with $c \in \mathbb{Q}$, then $\phi_2(f) = c \in \mathbb{Q}$, so ϕ_2 hits every element of $\mathbb{Q}$.

Thus $R/\ker \phi_2 = R/I \cong \operatorname{im} \phi_2 = \mathbb{Q}$.

Another approach would be to say there is 1 coset for each possible remainder after dividing by $x-2$. Then we see the only possible remainders are elements of $\mathbb{Q}$, (6)

so $R/I = \{ c + I : c \in \mathbb{Q} \}$, and then

you can check the obvious homomorphism

$$\begin{aligned} R/I &\longrightarrow \mathbb{Q} \\ c + I &\longmapsto c \end{aligned}$$

is actually an isomorphism.

#3a) totally straightforward, just like 2a, except you must satisfy <u>two</u> conditions to be in I.

Check: subset, $0 \in I$, closure under subtraction, and closure under mult.

#3b) No → $I \neq \langle (3x-y)(5x+2y) \rangle$. Let's see why: $(x-1)(x-2) \in I$, since it has $(1,3)$ and $(2,-5)$ as zeros. However, $(x-1)(x-2) \notin \langle (3x-y)(5x+2y) \rangle$. this is <u>not</u> obvious. You can give a term by term explanation of why any polynomial multiple $f(x,y)\cdot(3x-y)(5x+2y)$ must have at least one term with a y in it (as long as $f(x,y)$ is not the zero poly.). But there is a much slicker way:

<u>Lemma</u>: If $g(x,y) = f(x,y)\cdot(3x-y)(5x+2y)$ for some $f, g \in R$ (i.e. g is a polynomial multiple of $(3x-y)(5x+2y)$), and (a,b) is any zero of $(3x-y)(5x+2y)$, then (a,b) must also be a zero of ~~$g(x)$~~ $g(x,y)$, since $g(a,b) = f(a,b)\cdot 0 = 0$.

Thus, if we can find a zero of $(3x-y)(5x+2y)$ which is <u>not</u> also a zero of $(x-1)(x-2)$, this will show $(x-1)(x-2)$ is <u>not</u> a poly