# MATH 113, HOMEWORK #10

## Due Thursday, November 5

Remember, consult the Homework Guidelines for general instructions. Keep in mind that for us, rings are always required to have a multiplicative identity. This problem set consists of a mega problem to prep us for factor rings, particularly factor rings of polynomial rings (which is how we will eventually build finite fields other than $\mathbb{Z}_p$).

**GRADED HOMEWORK:**

1. For all problems, let $R = \mathbb{Z}_3[x]$.

   (a) Let $I$ be the subset of $R$ consisting of the zero polynomial, plus the polynomials for which every nonzero term has degree at least 2. (In other words, all polynomials for which the constant and linear terms are zero.) Prove that $I$ is an ideal of $R$. (We did not fully discuss the definition of ideals in class yet, but I trust you can look it up.)

   (b) The factor ring $R/I$ consists of (additive) cosets of the form $f(x) + I$, where $f(x) \in \mathbb{Z}_3[x]$. (For now you can take it as a given that this is indeed a ring, with well-defined coset addition and multiplication.) Prove that two polynomials in $\mathbb{Z}_3[x]$ belong to the same coset of $I$ if and only if they have the same remainder after division by the polynomial $x^2$, as prescribed by the Division Algorithm. *(Note that you are only being asked to prove this in this specific case, though the idea and proof certainly generalize to other polynomial rings and their ideals.)*

   (c) How many elements are in the ring $R/I$?

2. This problem is a continuation of problem 1, so $R = \mathbb{Z}_3[x]$, and $I$ is the same ideal as above.

   (a) Write out the multiplication table for $R/I$. Name each coset using its simplest element, e.g. $x + I$, not $(x + x^2 - 7x^{72}) + I$. (To save some writing, it is sufficient to fill in the upper right portion above and including the diagonal, since $R/I$ is a commutative ring in this case. Also, you need not include all your scratch work for the multiplication table.)

   (b) Determine whether each element of $R/I$ is a unit, a zero divisor, or neither.

3. This problem is also a continuation of problem 1, so $R = \mathbb{Z}_3[x]$, and $I$ is the same ideal as above. Prove that $R/I$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are isomorphic as additive groups, but not as rings.

   *Think about, but don't turn in: how would our answers to all three questions change if we had $R = \mathbb{Z}_4[x]$ instead of $\mathbb{Z}_3[x]$?*

**UNGRADED HOMEWORK:**

Note: Starred problems from this list are classic results you will almost certainly need to use again.

| Section | Problems |
|---|---|
| 22 | #6, 16, 17, 25, 27 |
| 23 | #9, 13, 14, 19, 20, 21, 29, 36* |
| 26 | #4, 7, 16, 18, 19, 30 |