HW #8 Solutions

#1) $\mathbb{Z}_8 \times \mathbb{Z}_6 \times \mathbb{Z}_4 / \langle(2,2,2)\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $H = \langle(2,2,2)\rangle$.
$G = \mathbb{Z}_8 \times \mathbb{Z}_6 \times \mathbb{Z}_4$.

Analysis of orders proof:

First, let's check every element in G/H has order at most 4: $4((a,b,c)+H)$

$$= (4a, 4b, 4c) + H$$
$$= (4a, 4b, 0) + H$$

$(4a, 4b, 0)$ can be any point in $\{0,4\} \times \{0,2,4\} \times \{0\}$, but all of these are in H:

$$H = \{\underline{(0,0,0)}, (2,2,2), \underline{(4,4,0)}, (6,0,2), \underline{(0,2,0)}, (2,4,2), \underline{(4,0,0)}, (6,2,2), \underline{(0,4,0)}, (2,0,2), \underline{(4,2,0)}, (6,4,2)\}$$

So no matter which coset $(a,b,c)+H$ is,

$$4((a,b,c)+H) = (0,0,0)+H,$$ so

every element has order at most 4.

G/H is an order 16 abelian group, since

$$\left|G/H\right| = \frac{|G|}{|H|} = \frac{8 \cdot 6 \cdot 4}{12} = 16.$$ By FTFGAG, G/H must be isomorphic to one of

$\mathbb{Z}_{16}$,
$\mathbb{Z}_8 \times \mathbb{Z}_2$,
$\mathbb{Z}_4 \times \mathbb{Z}_4$,
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,
or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

but we have already ruled out the first two.

Next, let's see we do have an element of order 4.

Claim: $\left|(1,0,0)+H\right| = 4$, since

$(1,0,0) \notin H$, $2\cdot(1,0,0) = (2,0,0) \notin H$, and
$3\cdot(1,0,0) = (3,0,0) \notin H$, but $4\cdot(1,0,0) = (4,0,0) \in H$.

Thus we can rule out the last option, and we are down to distinguishing between $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Note that $\mathbb{Z}_4 \times \mathbb{Z}_4$ has only 4 elements which are their own inverse (identity plus the order 2 elements): those in $\{0,2\} \times \{0,2\}$.

But $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has eight elements which are their own inverse: those in $\{0,2\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

If we can find at least 5 elements in G/H which are their own inverses, we will have shown $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. (Note we must be care ful not to pick reprecentatives from the same coset in our work).

	$(a,b,c)+H$	$2((a,b,c)+H)$
i)	$(0,0,0)+H$	$(0,0,0)+H$ ✓
ii)	$(1,1,1)+H$	$(2,2,2)+H = id$ ✓
iii)	$(0,1,0)+H$	$(0,2,0)+H = id$ ✓
iv)	$(1,0,1)+H$	$(2,0,2)+H = id$ ✓
v)	$(2,0,0)+H$	$(4,0,0)+H = id$ ✓

Note $(1,1,1)-(0,1,0) \notin H$, so ii) and iii) are different cosets.

Similarly
$(1,1,1) - (1,0,1) \notin H$
$(1,0,1) - (0,1,0) \notin H$,
etc.

Proof by 1^{st} $\cong$ som thm.

Let $\varphi: \mathbb{Z}_8 \times \mathbb{Z}_6 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_8 \times \mathbb{Z}_6 \times \mathbb{Z}_4$

$$(a, b, c) \longmapsto (2a + 2c, 3b, 2c)$$

We claim ~~then~~ • φ is a well-def. homomorphism

• $\ker \varphi = \langle (2, 2, 2) \rangle$

• $\operatorname{im} \varphi \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

• Perhaps the trickiest part of this is getting φ to be well-defined. To do so, we need a previous HW result (which many people did not explore). A ~~homomorphism~~ potential* map* $\mathbb{Z}_a \to \mathbb{Z}_b$

$$x \longmapsto cx$$

will be well-defined iff $b \mid ac$ (or you could use $b \mid \operatorname{lcm}(ac)$), & then it will be a homomorphism

For our φ above this means if we build φ from nice "linear" pieces (which is the only sensible* way to get a homomorphism from $\mathbb{Z}_n$'s $\to \mathbb{Z}_m$'s.), then our map must use only multiples of

→ a, $4b$, and $2c$ in the first coord,

→ $2a$, b, and $3c$ in the second, and

→ a, $2b$, and c in the third.

Our given map fits this, so it will be a homomorphism.

- Next let's check the kernel.

Definitely $(2,2,2) \in \ker \varphi$, as

$$(2,2,2) \longmapsto (2\cdot 2 + 2\cdot 2,\ 2\cdot 3,\ 2\cdot 2)$$
$$= (0,0,0) \in \mathbb{Z}_8 \times \mathbb{Z}_6 \times \mathbb{Z}_4.$$

By the homomorphism property (repeated application), we can see that all of $\langle (2,2,2) \rangle \in \ker \varphi$.

Rather than directly checking nothing else can be in $\ker \varphi$, let's see that the image is large enough.

- <u>Claim</u>: $\operatorname{im}\varphi = \{0,2,4,6\} \times \{0,3\} \times \{0,2\}$.

Clearly, choosing b and c as you wish will get all possibilities for the second and third coordinates.

If $2c = 0$ in $\mathbb{Z}_4$, we can clearly get all possibilities for the first and second coords.

If $2c = 2$ in $\mathbb{Z}_4$, it takes more work, but we can still get all the combinations. Eg if $c=1, a=2, 2a+2c=6$
$c=1, a=3, 2a+2c=2$
etc.

Thus $\operatorname{im}\varphi$ has 16 elements, which means the kernel must have 12, and we have already exhibited 12. So $\ker\varphi = \langle(2,2,2)\rangle$, $\operatorname{im}\varphi \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and thus by the first Isomorphism thm,

$$G/H \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

⑦

#2a) T is <u>not</u> a subring of $\mathbb{R}$, as it is not closed under mult.

Let $x = \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{bmatrix}$ $y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

then $xy = \begin{bmatrix} 1+\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$

So $x, y \in T$ (since trace x = trace $y = 2) \in \mathbb{Q}$.,

but $xy \notin T$ (since trace $xy = 2+\sqrt{2} \notin \mathbb{Q}$,
so T is not a (sub) ring.

#2b) D is <u>not</u> a sub ring of $\mathbb{R}$, as it is not closed under add/subt.

Let x and y be the same matrices as above, so $\det x = \det y = 1 \in \mathbb{Q}$.

Then $x+y = \begin{bmatrix} 2 & \sqrt{2} \\ 1 & 2 \end{bmatrix}$; and $\det(x+y) = 4-\sqrt{2}$. $\notin \mathbb{Q}$

thus $x, y \in D$, but $x+y \notin D$, so

D is not a (sub)ring.

#2c) This one can be a little informal.

(ok to use linear alg knowledge.)

L <u>is</u> a subring of R

- By definition $L \subseteq R$.
- L contains our identies, as both I and $0 \cdot I$ are lower triangular.
- Suppose x and y are two lower triangular matrices. then their difference $x-y$ is also lower triangular.
- Similarly, so is the product xy.

thus by the subring criterion, L is a subring of R.

#3b a) Totally straight forward → use the subring criterion, as in # 2b.

- Clearly $S \subseteq R$, by def.
- Since constant terms are even degree (deg 0), $0, 1 \in S$.
- Let ~~a(x), b(x), a(x) can~~ $p(x)$ and $q(x) \in S$. Then their difference will only have even degree terms,
- and so will their product.

Thus S is a subring of R

#3b) First we will check the two homomorphism properties. Let $p(x), q(x) \in R$.

Additive: Then $\varphi(p(x) + q(x)) = \varphi((p+q)(x))$

$$= ((p+q)(0), (p+q)(1)) = (p(0)+q(0), p(1)+q(1))$$

$$= (p(0), p(1)) + (q(0), q(1)) = \varphi(p(x)) + \varphi(q(x)).$$

with the portion marked by the arrow due to the pointwise addition of functions (& thus polynomials).

Mult: $\varphi(p(x)\cdot q(x)) = \varphi((pq)(x)) = ((pq)(0), (pq)(1))$

$= (p(0)\cdot q(0), p(1)\cdot q(1)) = (p(0), p(1))\cdot(q(0), q(1$

$= \varphi(p(x))\cdot\varphi(q(x))$.

Thus φ is a homomorphism.

Notice $\ker\varphi = \{p(x)\in R \mid (p(0), p(1)) = (0,0)\}$.

i.e. those polynomials with $p(0)=0$ AND $p(1)=$

Thus $\ker\varphi$ consists exactly of those polynomials that have roots at $x=0$ and $x=1$, or equivalently have factors of x and $(x-1)$

$$\ker\varphi = \{x(x-1)\cdot f(x) : f(x)\in R\}.$$