MATH 113, HOMEWORK #7

Due Thursday, October 8

Remember, consult the Homework Guidelines for general instructions. From now on, all work with permutations should be done in disjoint cycle notation. As before, any dihedral group problems should use our r, s notation. This is the last homework due before the midterm.

GRADED HOMEWORK:

1. Let F denote the additive group of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
   (a) Let $\varphi : F \to \mathbb{R}$ be defined by $f \mapsto \int_0^1 f(x)dx$ for all $f \in F$. Prove that φ is a group homomorphism.
   (b) Find the kernel K of φ – give algebraic and geometric descriptions. (Feel free to use prior calculus knowledge.)
   (c) Describe the coset $x + K$ of the kernel, with both algebraic and geometric descriptions. Is there a "nicer" representative of this coset?
   (d) Note that the cosets of K are in bijection with $\mathbb{R}$. Does Lagrange's Theorem apply here (to K and F)? Explain very briefly.
2. You may use part (a) to solve part (b) below.
   (a) Suppose G is a finite group of order m. Prove that $g^m = e$ for all $g \in G$.
   (b) Suppose G is a finite group, and $N \trianglelefteq G$. If there are k cosets of N in G, prove that $g^k \in N$ for all $g \in G$.
3. This problem will deal with the group $G = D_4 \times S_3$.
   (a) How many elements of each order do the groups D_4 and S_3 have? Using this info, determine how many elements of each order G has. (Of course, it would be extremely tedious to compute all 48 orders one by one. You should NOT be doing that. Perhaps organize your work in a table.)
   (b) Find all subgroups of G that are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Be sure to explain why you have found all of them.

Don't turn the following problems in – they are just extra practice before the midterm.

* Suppose a, b, and c are integers, with $a, b \geq 2$. Find necessary and sufficient conditions for the map $\varphi : \mathbb{Z}_a \to \mathbb{Z}_b$ sending $\overline{x} \mapsto \overline{cx}$ to be a homomorphism. Justify your answer. (This one will likely take some experimentation before you find a solution. For example, you can verify that $c = 0$ always works, no matter what a and b are.)

* This problem shows that knowing the isomorphism types of two of G, H, and G/H is not sufficient to understand the isomorphism type of the third.
   (a) In $G = D_6$, find two nonisomorphic subgroups H and K so that $G/H \cong G/K$.
   (b) In $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, find two subgroups H and K so that $H \cong K$, but $G/H \ncong G/K$.
   (c) Find two groups G_1 and G_2 and corresponding subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$ so that $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2$, but $G_1 \ncong G_2$.

UNGRADED HOMEWORK:

Starred problems from this list are classic results you will almost certainly need to use again. Ignore the external direct product business in TF.

Section	Problems
13	#32, 36, 44*, 45*, 47, 51
14	#23, 30, 31, 32, 35, 40
15	#19