Math 113 HW 7 Solutions

#1a) Nothing tricky here - just use calculus facts.

Pf Suppose $f, g \in F$.

Then $\varphi(f+g) = \int_0^1 (f+g)(x)\,dx = \int_0^1 f(x)+g(x)\,dx$

$= \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$

$= \varphi(f) + \varphi(g)$, so

φ is a homomorphism.

#1b) $K = \ker\varphi = \{ f \in F : \int_0^1 f(x)\,dx = 0 \}$ (algebraic)

$= \{$ functions in F with signed area $= 0$ between $x=0$ and $x=1 \}$.

ie (area above x-axis) − (area below x-axis)

(geometric)

#1c) $x + K = \left\{ f \in F : \int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2} \right\}$

$= \left\{ \begin{array}{l} \text{functions in } F \text{ with signed area} \\ \frac{1}{2} \text{ between } x=0 \text{ and } 1 \end{array} \right\}.$

A nicer name would be $\frac{1}{2} + K$, since the $\frac{1}{2}$ function $\left(\text{i.e. } \begin{array}{l} f : \mathbb{R} \to \mathbb{R} \\ x \mapsto \frac{1}{2} \end{array} \right)$ is even simpler than ~~the~~ the x function.

#1d) Just as $\frac{1}{2} + K = x + K$, every function in F belongs to ~~some~~ exactly one coset $r + K$ with $r \in \mathbb{R}$, since each will have the same signed area as one constant function.

Lagrange's thm is irrelevant, since $|F|$ is infinite.

#2a) Proof
Let $g \in G$. By Lagrange's thm*, the order of g divides m. I.e. $|g| = k$ where k is a divisor of m, so $m = kl$ f.s. $l \in \mathbb{Z}^+$.

Since $|g| = k$, we know $g^k = e$. But then $(g^k)^l = e^l$, so

$g^{kl} = e$, i.e.

$g^m = e$.

* technically a corollary of Lagrange, but ok to call this Lagrange's thm too.

#2b) Proof: By the given statements, we see

G/N is a group, and $|G/N| = k$.

Let $g \in G$. then $gN \in G/N$, so by part (a), $(gN)^k = eN$, ie $g^k N = eN$.

this implies $g^k \in N$.

#3d) D_4 has the following # of elements of each order:

order	# elements	
1	1	(id only)
2	5	(4 refl, 180° rot)
4	2	(90° & 270° rot)
8	0	(not cyclic)

For S_3 we have

order	# elements	
1	1	(id)
2	3	(2-cycles)
3	2	(3-cycles)
6	0	(not cyclic)

In the product $D_4 \times S_3$, an element has

$$|(a,b)| = \operatorname{lcm}(\underset{\text{computed in } D_4}{|a|}, \underset{\text{computed in } S_3}{|b|})$$

Thus we could fill in the following table to get a quick count of the # of elements of each order.

$D_4 \downarrow$ \ $S_3 \rightarrow$	1 (1 elt)	2 (3 elts)	3 (2 elts)
1 (1 elt)	lcm(1,1) = 1 (1 elt)	lcm = 2 (3 elts)	lcm = 3 (2 elts)
2 (5 elts)	lcm = 2 (5 elts)	lcm = ~~4~~ 2 (15 elts)	lcm = 6 (10 elts)
4 (2 elts)	lcm = 4 (2 elts)	lcm = 4 (6 elts)	lcm = 12 (4 elts)

totalling, we get in $D_4 \times S_3$:

order	# elements
1	1
2	3 + 5 + 15 = 23
3	2
4	2 + 6 = 8
6	10
12	4

total 48 ✓

#36) Let's consider our possibilities for subgroups $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Each one must consist of the identity $(e_{D_4}, e_{S_3}) = (e,e)$ plus 3 more elements of order 2 (of which we have 23). We can't pick just any subset of 3, or closure might fail (or abelian-ness, etc).

We have shown that if $A \leq D_4$ and $B \leq S_3$, then $A \times B \leq D_4 \times S_3$ (well, more generally), so one option is to take the product of an order 2 subgroup of D_4 and an order 2 subgroup of S_3.

For example, $\langle r^2 \rangle \times \langle (1,2) \rangle$.

We will get 15 such subgroups $A \times B$, where ~~A is $\langle r^2\rangle, \langle s\rangle, \langle sr\rangle, \langle sr^2\rangle, \langle sr^3\rangle$~~

A is $\langle r^2 \rangle, \langle s \rangle, \langle sr \rangle, \langle sr^2 \rangle$, or $\langle sr^3 \rangle$

and B is $\langle (1,2) \rangle$, $\langle (1,3) \rangle$, or $\langle (2,3) \rangle$.

However, this is not everything, as D_4 alone has multiple subgroups $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Namely $\langle r^2, s \rangle$, or $\langle r^2, sr \rangle$

thus $\{e, r^2, s, sr^2\} \times \{e_{S_3}\}$

and $\{e, r^2, sr, sr^3\} \times \{e_{S_3}\}$

are two more subgroups $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(Note we can't do this the other way around since S_3 does not have subgroups $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.)

How can we check we have everything?

First notice if our subgroup has any element with a (2-cycle) in the second coordinate, then every other element in our subgroup must have either the identity or the same 2-cycle in the second coordinate. (Otherwise we would have a non abelian group, since no two 2-cycles in S_3 commute)

Continuing this case, which elements could appear in the first coordinate? Note you can only have one non-identity element in the 1st coordinate, or we would get too many elements. Our list of 15 subgroups covers all of these.

Then it remains to check D_4 has only 2 subgroups $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which we have essentially done in previous HW.