HW #6 Solutions

#1a) This is false. You can show that $G \times H$ has at least kl subgroups by showing each $A \times B$ with $A \leq G$ and $B \leq H$ is a subgroup of $G \times H$, but there may be other subgroups.

For example, if $G = \mathbb{Z}_4$ and $H = \mathbb{Z}_2$, then G has 3 subgroups (1 each of order 1, 2, 4) and H has 2 subgroups. As mentioned above, all $A \times B$ will be subgroups:

$\{0\} \times \{0\}$
$\{0\} \times \mathbb{Z}_2$
$\langle 2 \rangle \times \{0\}$
$\langle 2 \rangle \times \mathbb{Z}_2$
$\mathbb{Z}_4 \times \{0\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2$

However, there is at least one more: $\langle (1,1) \rangle$, which is an order 4 subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_2$, not equal to any of the 6 we have listed.

#1b) this is false. It is easiest to find a counter example for $n=3$ or $n=4$.

Let's look at D_3 – could it be ~~isom to~~ isomorphic to a product of an order 3 group and an order 2 group? All groups of order 3 are abelian (actually cyclic, since 3 is prime) and so are all groups of order 2. Thus, any such product will be abelian. Since D_3 is not abelian, it cannot be isomorphic to such a product.

(For D_4: we have determined in previous HW that all order 4 subgroups are isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, which are both abelian. The rest of the argument is the same.)

#2a)

$eH = \{e, r^2, r^4\} = He$

$rH = \{r, r^3, r^5\} = Hr$

$sH = \{se, sr^2, sr^4\} \overset{?}{=} \{es, r^2s, r^4s\} = Hs$

$= \{s, sr^2, sr^4\} = \{s, sr^4, sr^2\} = Hs$

$(sr)H = \{sr \cdot e, sr \cdot r^2, sr \cdot r^4\} \overset{?}{=} \{esr, r^2sr, r^4sr\} = H(sr)$

$= \{sr, sr^3, sr^5\} = \{sr, sr^5, sr^3\} = H(sr)$ ✓

The left and right coset decompositions coincide.

#2b)

coset mult	eH	rH	sH	$(sr)H$
eH	eH	rH	sH	srH
rH	rH	eH	srH	sH
sH	sH	srH	eH	rH
$(sr)H$	srH	sH	rH	eH

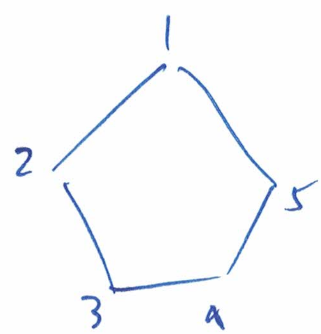
$rH \cdot sH$
$= rsH = sr^3H$
$= srH$

Might include some work like this.

#2b cont) Our group (which we now know to call D_6/H) is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (or the Klein 4-group), since it is an order 4 group in which every element is its own inverse.

#3a) this one is super hard unless you make the connection to D_5.

Take $\langle (1,2,3,4,5),\ \cancel{(2,3)(4,5)}\ (2,5)(3,4) \rangle$.



As in our D_6 example handout, we can find an isomorphism with a subgroup of S_5.

$$\left.\begin{array}{l} r \longleftrightarrow (1,2,3,4,5) \\ \text{and } s \longleftrightarrow (2,5)\cdot(3,4) \end{array}\right\} \text{How } r, s \text{ permute the vertices.}$$

Thus, the subgroup generated by these two elements will be of order 10.

(Note: S_5 does <u>NOT</u> have elements of order 10, so you can't get a cyclic example here.)

Note 2: Also ok to list the 10 elements:

- five 5-cycles
- ~~for~~ four pairs of disjoint 2-cycles
- + e

~~#3b)~~

#3b) Here we can find a cyclic subgroup, even though S_5 doesn't have 6-cycles.

Take $\sigma = (1,2)(3,4,5)$. Then

$|\sigma| = 6 = \text{lcm}(2,3)$

$=$ lcm of cycle lengths when in <u>disjoint</u> cycle notation.

Thus $\langle (1,2)(3,4,5) \rangle$ works.

Note: Here you may have gotten lucky*, as

$$\langle (1,2)(3,4,5) \rangle = \langle (1,2), (3,4,5) \rangle$$

↗ one generator   two generators

* Not exactly lucky → it's basically because $\gcd(2,3)=1$.

But in general you need to be more careful.

Eg. $\langle (1,2)(3,4) \rangle \neq \langle (1,2), (3,4) \rangle$

$\cong \mathbb{Z}_2$ $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

#3c) Many examples here, eg:

- S_7 (really any S_n with $n \geq 7$)
- $GL(3, \mathbb{C}) \times \mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_7$
- (any nonabelian group) $\times \mathbb{Z}_6 \times \mathbb{Z}_7$
- $D_3 \times D_7 \times \mathbb{Z}_2$

etc.

#3d) Let x be a nonidentity element in $U_{11} \leq \mathbb{C}$ (i.e. a primitive 11^{th} root of unity).

Then $\left\langle \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \right\rangle$ has 11 elements

So does $\left\langle \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$ or $\left\langle \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \right\rangle$.

(I think these might be the only really simple ones.)

If you know about rotation matrices, you could also take

$$\left\langle \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\rangle \text{ where } \theta = \frac{2\pi}{11} \text{ radians.}$$

which actually sits in $GL(2, \mathbb{R})$.