113 HW #5 Solutions

#1a) From the ungraded HW, you should have discovered that the order of a permutation in disjoint cycle notation is the lcm of its cycle lengths. So, a permutation with a 2-cycle and a 5-cycle will work here, & S_7 is big enough we can make those disjoint.

Eg. $x = (1,2)(3,4,5,6,7)$ ← generator

$x^2 = (3,5,7,4,6)$

$x^3 = (1,2)(3,6,4,7,5)$ ← generator

$x^4 = (3,7,6,5,4)$

$x^5 = (1,2)$

$x^6 = (3,4,5,6,7)$

$x^7 = (1,2)(3,5,7,4,6)$ ← generator

$x^8 = (3,6,4,7,5)$

$x^9 = (1,2)(3,7,6,5,4)$ ← generator.

$x^{10} = e$

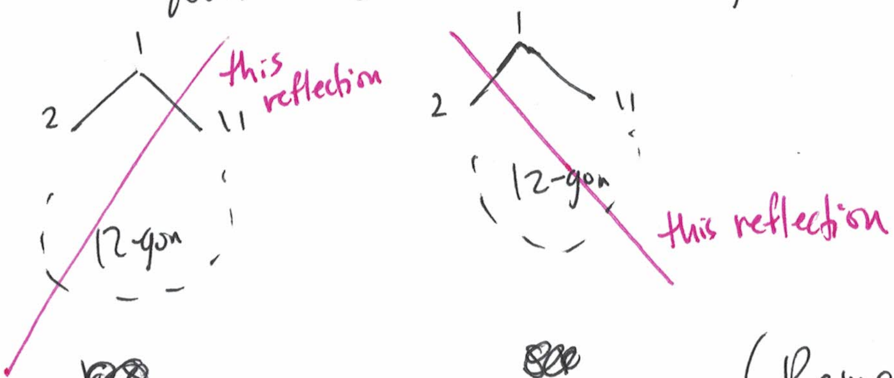
$G = \langle (1,2)(3,4,5,6,7) \rangle$ is a cyclic group of order 10, so it is isomorphic to $\mathbb{Z}_{10}$ and has the same number of generators, i.e. 4, since 1, 3, 7, 9 are relatively prime to 10, and 0, 2, 4, 5, 6, 8 are not. The generators are exactly x^1, x^3, x^7, and x^9.

#1b) By ungraded HW from Section 6, an isomorphism between cyclic groups can be completely determined by the image of a single generator. I.e., if $\phi : G \longrightarrow U_{10}$ is an isomorphism, $G = \langle x \rangle$, and we know which element $\phi(x)$ is, then the homomorphism property uniquely determines where the other powers of σ must map to. Now x can map to any of U_{10}'s four generators, but nowhere else (since it must map to an element of ~~ord~~ order 10).

(It's #44 in Sec. 6.)

Thus, there are 4 isomorphisms.

#2a) • D_{12} is non abelian since r & s don't commute. (Many other pairs possible.)



$sr \neq rs$ (Remember: work $R \to L$.)

• S_4 is nonabelian since $(1,2)$ and $(2,3)$ don't commute (again, many pairs possible):

$$(1,2)(2,3) = (1,2,3)$$
$$\neq$$
$$(2,3)(1,2) = (1,3,2)$$

• Probably the easiest way to check they are not isomorphic is to compare elements of a certain order. I'll give a complete list on the next page, but you only need to pick one order to work with.

order of element	D_{12} elements (how many)	S_4 elements (how many)
1	e — 1	e — 1
2	all reflections & r^6 — 13	transp (a,b) $\binom{4}{2}=6$ and pair of transp. (disjoint) $(a,b)(c,d)$ $\binom{4}{2,2}=3$ 9
3	r^4, r^8 — 2	3-cycle (a,b,c) $\binom{4}{3}\cdot 2$ — 8
4	r^3, r^9 — 2	4-cycle (a,b,c,d) — 6 $(1, \underset{\uparrow}{\_}, \underset{\uparrow}{\_}, \underset{\uparrow}{\_})$ 3 choices, 2 choices, 1 choice
6	r^2, r^{10} — 2	0
8	0	0
12	r, r^5, r^7, r^{11} — 4	0
24	0	0
	total 24	total 24

#2b) Yes, actually several choices.

Easiest $= \{e, r^6, s, sr^6\}$

$\circ$	e	r^6	s	sr^6
e	$\boxed{e}$	r^6	s	sr^6
r^6	r^6	$\boxed{e}$	sr^6	s
s	s	sr^6	$\boxed{e}$	r^6
sr^6	sr^6	s	r^6	$\boxed{e}$

row $\circ$ col

order 4, e's on diag $\Rightarrow \cong$ Klein 4

Could also do: $\{e, r^6, sr, sr^7\}$

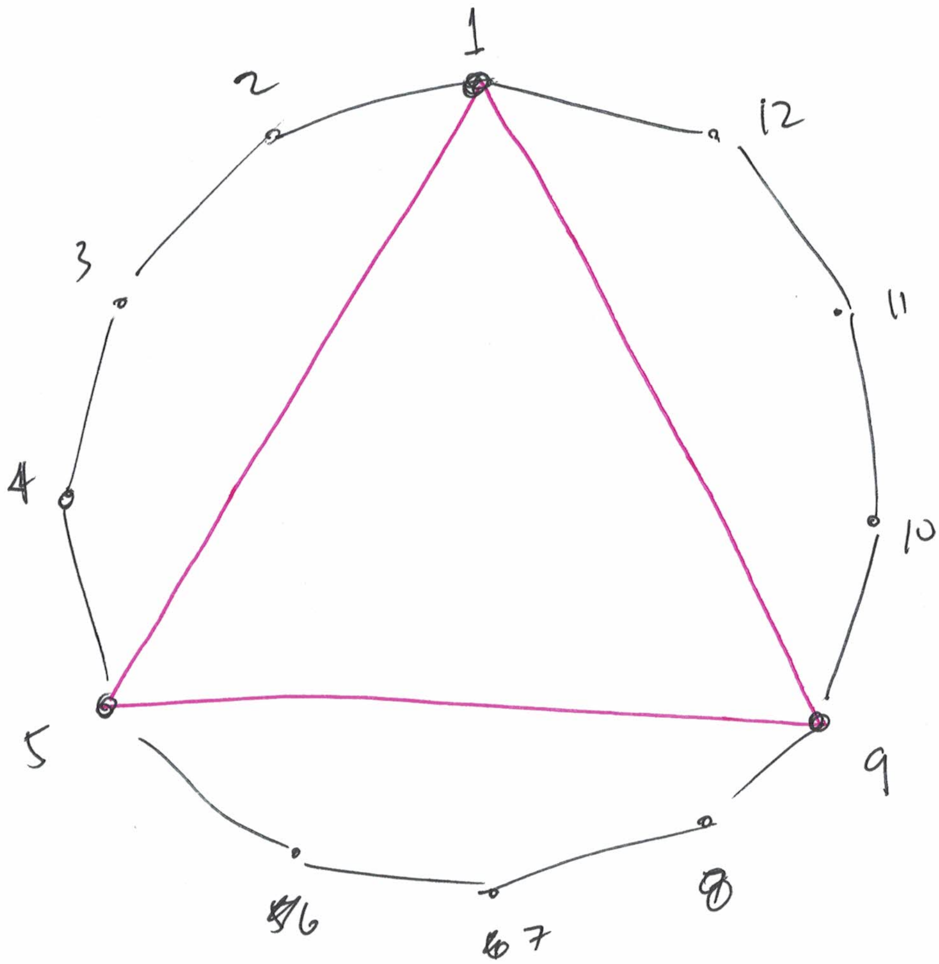
or $\{e, r^6, sr^2, sr^8\}$

or $\{e, r^6, sr^3, sr^9\}$

or $\{e, r^6, sr^4, sr^{10}\}$

or $\{e, r^6, sr^5, sr^{11}\}$.

#2c) Take $H = \langle r^4, s \rangle = \{e, r^4, r^8, s, sr^4, sr^8\}$.



Every symmetry in D_{12} built using r^4 and s is also a symmetry of the pink Δ, and it covers all 6 such symmetries.

Thus $H \cong D_3$, and as the book has discussed, $D_3 \cong S_3$, so $H \cong S_3$.

#3a) $L = \langle r^2, s\rangle \cong D_3$. Same geom. reasoning as #2c.

#3b) $K = \{e, (1,8)(2,9), (3,7)(5,6), (1,8)(2,9)(3,7)(5,6)\}$.

Next, K is a subgroup of A_{10}, since all of its elements are products of an even # of transpositions.

Finally $K \cong$ Klein 4 group V, since K is an order 4 group that is not cyclic, since all elements are their own inverses.

(Also ok to show the group table as justification.)