MATH 113, HOMEWORK #4

Due Thursday, September 17

Remember, consult the Homework Guidelines for general instructions.

GRADED HOMEWORK:

1. This problem deals with subgroups of $GL(2, \mathbb{R})$.

   (a) Prove that the set
   $$H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$
   is NOT a subgroup of $GL(2, \mathbb{R})$. (You may use any standard linear algebra facts about matrices as long as you state them.)

   (b) Find an infinite subset of the set H defined in part (a) which is a cyclic subgroup of $GL(2, \mathbb{R})$. Prove that your answer works.

   (c) Find an infinite subset of the set H defined in part (a) which is a noncyclic subgroup of $GL(2, \mathbb{R})$. Prove that your answer works.

2. Here we will think about products of groups.

   (a) Prove that if G and H are groups, then $G \times H$ is a group. Specifically, it consists of the set $G \times H = \{(g, h) : g \in G, h \in H\}$ with the binary operation $*$ defined componentwise, i.e. $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$ by definition (for all ordered pairs in $G \times H$), where the multiplication in the first component is computed using G's binary operation, and the multiplication in the second component is computed using H's binary operation. Note that this is not a subgroup of any of our known groups, so you will need to check associativity.

   (b) Consider the example $\mathbb{C}^* \times \mathbb{C}^*$ (with typical complex multiplication as the binary operation in each component). Find two subgroups of order 8 in $\mathbb{C}^* \times \mathbb{C}^*$ – one which is cyclic and one which is not. (Use our exponential/polar form if necessary.) Be sure you justify your answers.

   (c) *Think about, but don't turn in: by experimenting with group tables, convince yourself that every group of order 4 is either cyclic or isomorphic to the Klein 4-group.*

3. This problem is about cyclic groups and generators.

   (a) In class, we saw that $\mathbb{Z}_6$ has exactly two different generators. Find two choices of n so that $\mathbb{Z}_n$ has exactly four different generators. Justify your answer.

   (b) Which of the following are cyclic groups? If the group is cyclic, find one generator for it, and list the group elements in the order they are generated. (For example, for $\mathbb{Z}_6$ generated by $\overline{1}$, I would list $\{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{0}\}$.) If the group is not cyclic, explain why.

   - $\mathbb{Z}_2 \times \mathbb{Z}_3$
   - $\mathbb{Z}_2 \times \mathbb{Z}_4$

UNGRADED HOMEWORK:

Section	Problems
6	#32, 33, 34, 35, 36, 37, 40, 43, 44, 46, 51