HW #4 Solutions

#1a) H does not contain the inverses of all of its elements. For example, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \in H$, but $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \notin H$.

#1b) Almost any cyclic subgroup $\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\rangle$ of H will work.

An easy one to check is

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n : n \in \mathbb{Z} \right\}$$

$$= \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

#1c) A couple reasonable choices are

$K = \{ M \in H : \det M = 1 \}$ or

$L = \{ M \in H : \det M = \pm 1 \}$.

Showing you have a subgroup here is

completely straightforward. Use these linear algebra facts: $\det(AB) = \det A \cdot \det B$

and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The hard part is to see that K or L is not cyclic. We can tell it's infinite, since it contains our subgroup from part b). Thus, if it were cyclic, it would have to be isomorphic to $\mathbb{Z}$.

But note that every nonidentity element in $\mathbb{Z}$ generates an infinite subgroup / i.e. has infinite order.

However, K and L both have nonidentity elements of finite order, e.g.

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has order 2.

Since $\mathbb{Z}$ has no order 2 element, our subgroup can't be isomorphic to $\mathbb{Z}$ and thus can't be cyclic.

#2a) This is covered in more generality at the beginning of Section 11.

#2b) To find a cyclic group of order 8, we need an element of order 8.

Some simple choices are

$\langle (e^{2\pi i/8}, 1) \rangle = U_8 \times \{1\}$,

$\langle (1, e^{2\pi i/8}) \rangle = \{1\} \times U_8$,

$\langle (e^{2\pi i/8}, e^{2\pi i/8}) \rangle$ (Not a product group.

Actually, anything of the form

$\langle (\text{generator of } U_8, \text{ any element in } U_8) \rangle$

will work; same with coordinates reversed.

The easiest noncyclic subgroups of order 8 are $U_4 \times U_2$ OR $U_2 \times U_4$.

(Actually these may be the only ones.)

Simple counting shows you have 8 elements, and part a) proves we have a subgroup, since $U_2 \leq \mathbb{C}^*$ and $U_4 \leq \mathbb{C}^*$,

(Okay to use $U_2 = \{1, -1\}$
$U_4 = \{1, -1, i, -i\}$.)

We need to show this group is not cyclic. Say we are using $U_4 \times U_2$ (the other proof just has the coordinates flipped). To do so, I'll show each element has order ≤ 4.

Let $(a, b) \in U_4 \times U_2$.

Then $(a, b)^4 = (a^4, b^4)$, by the definition of the binary operation for $\mathbb{C}^* \times \mathbb{C}^*$.

Notice $a^4 = 1$ for all $a \in U_4$

and $b^4 = (b^2)^2 = 1$ for all $b \in U_2$,

since these coords are fourth & square roots of unity respectively.

Thus our subgroup has no element of order 8, the size of the subgroup, so it can't be cyclic.

Note: Also ok to individually check the elements aren't generators this time, but you should probably learn the technique above.

#3a) A couple choices are $n=5$, $n=10$, $n=12$.

These all have exactly 4 positive integers k so that $\gcd(k,n)=1$ and $1 \le k \le n$.

$n=5$	$n=10$	$n=12$
$k=1,2,3,4$	$k=1,3,7,9$	$k=1,5,7,11$

#3b) $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic. Example generators:

$$\langle(1,1)\rangle = \{(1,1),(0,2),(1,0),(0,1),(1,2),(0,0)\}.$$

OR

$$\langle(1,2)\rangle = \{(1,2),(0,1),(1,0),(0,2),(1,1),(0,0)\}$$

3b) cont.

- $\mathbb{Z}_2 \times \mathbb{Z}_4$ is NOT cyclic.

  You can use the same argument I used in # 2b, or perhaps check the various cyclic subgroups:

$$\langle (0,0) \rangle = \{(0,0)\}.$$

$$\langle (1,0) \rangle = \{(0,0), (1,0)\}.$$

$$\langle (0,1) \rangle = \langle (0,3) \rangle = \{(0,0), (0,1), (0,2), (0,3)\}.$$

$$\langle (0,2) \rangle = \{(0,0), (0,2)\}.$$

$$\langle (1,1) \rangle = \{(0,0), (1,1), (0,2), (1,3)\}$$

$$= \langle (1,3) \rangle$$

$$\langle (1,2) \rangle = \{(0,0), (1,2)\}.$$