MATH 113, HOMEWORK #3
Due Thursday, September 10

Remember, consult the Homework Guidelines for general instructions.

GRADED HOMEWORK:

Note: Starting with this HW, feel free to stop writing the $*$ for arbitrary/unknown binary operations. (But definitely leave in $+/-$ if your operation is addition, of course.) You can just write ab in place of $a * b$ and a^n for $a * \cdots * a$ (n copies).

1. Let $G = \langle \mathbb{R}^3, + \rangle$, the group of real 3-tuples with componentwise addition.
   (a) What geometric object is the set
   $$H = \{(x, y, z) \in \mathbb{R}^3 : 3(x-1) + 2(y+5) - (z+1) = 6\}?$$
   Prove that H is a subgroup of G.
   (b) The cyclic subgroup generated by $(0, 0, 0)$ is the trivial subgroup of G, but other cyclic subgroups are more interesting. Let K be the cyclic subgroup generated by a nonidentity element (a, b, c). Give algebraic and geometric descriptions of K.
   (c) Let L be the line through the origin and the point $(2, 3, 5)$. Is L a subgroup of G? If so, is it a cyclic subgroup of G? (It is easiest to check if you come up with an algebraic description of L.) Be sure to justify your answers.

2. Let $G = GL(3, \mathbb{R})$, the group of 3×3 invertible matrices under multiplication.
   (a) Let L be the set of lower triangular real 3×3 matrices with ones on the diagonal, i.e.
   $$L = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$
   Prove that L is a subgroup of G.
   (b) Note that G is not abelian, but it does have abelian subgroups, such as the trivial subgroup $\{I_3\}$. Find two nontrivial abelian subgroups H and K of G which are not isomorphic to each other. Justify all parts (subgroup, abelian, not isomorphic to each other).

3. Mini proofs – these should be short, but be sure to write them well! Let the function $\phi : G \to H$, where G and H are groups, be a group homomorphism, i.e. $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. (This is the same definition we had for binary structures, except we know the domain and codomain are both groups, and the equation has the stars dropped.)
   (a) Prove that if a and b are inverses of each other in G, then $\phi(a)$ and $\phi(b)$ are inverses of each other in H. Conclude that if an element x is its own inverse in G, then $\phi(x)$ is its own inverse in H.
   (b) If G is abelian, prove that $\phi(g_2)\phi(g_1) = \phi(g_1)\phi(g_2)$ for all $g_1, g_2 \in G$. Does this prove that H is also abelian? Why or why not?

UNGRADED HOMEWORK:

Note: Starred problems from this list are classic results you will almost certainly need to use again.

Section	Problems
4	20, 32, 33, 41
5	20, 28, 30, 31, 34, 36, 39, 41*, 43, 51, 54*