# HW #3 Solutions

#1a) $H$ is a plane (standard 53 stuff).

Let's check $H \leq G$: Note $H = \{(x,y,z) \in G : 3x+2y-z=0\}$.

- $H \subseteq G$ by definition.
- Identity. Check $(0,0,0) \in H$.

  $3(0-1) + 2(0+5) - (0+1) = -3+10-1 = 6$,

  so $(0,0,0) \in H$.
- Inverses. Suppose $(x,y,z) \in H$, i.e. — using the simplified condition to belong to $H$ —

  ☆ $3x+2y-z=0$. WTS: $(-x,-y,-z) \in H$.

  We have $3(-x) + 2(-y) - (-z)$

  $$= -(3x+2y-z) = -0, \text{ using ☆.}$$
  $$= 0,$$

  so $H$ contains inverses of its elements.
- Closure. Suppose $(x_1, y_1, z_1), (x_2, y_2, z_2) \in H$.

  WTS: $(x_1, y_1, z_1) + (x_2, y_2, z_2) \in H$.

  We know $3x_1 + 2y_1 - z_1 = 0$ and

  $3x_2 + 2y_2 - z_2 = 0$. Adding these:

  $$3(x_1+x_2) + 2(y_1+y_2) - (z_1+z_2) = 0$$

Thus $(x_1+x_2,\ y_1+y_2,\ z_1+z_2) = (x_1,y_1,z_1)+(x_2,y_2,z_2) \in H$, and $H$ is closed under addition.

These four conditions imply $H \leq G$.

#1b) The cyclic subgroup

$$K = \langle (a,b,c) \rangle = \{ n(a,b,c) : n \in \mathbb{Z} \}$$

consists of equally spaced points along the line through $(0,0,0)$ with direction vector $\langle a,b,c \rangle$. Note it is NOT the full line.

#1c) $L = \{ t\cdot(2,3,5) : t \in \mathbb{R} \}$. $L$ is a subgroup of $G$. (this is a standard check, nothing crazy here; you need to show it.) $L$ is not cyclic, as it consists of uncountably many points, and cyclic groups have at most finitely many. (OR: $L$ is not of the form we discovered in #1b, so it can't be cyclic.)

#2a) This is again a standard check, summarized here.

- $L \subseteq G$ since every matrix in $L$ has determinant 1 and is thus an invertible $3\times 3$ matrix.

- Identity: Clearly $I_3 \in L$, with $a=b=c=0$.
- Inverses: Let $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \in L$. Then

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}.$$

As $-a$, $ac-b$, $-c$ are all in $\mathbb{R}$, this inverse is also in $L$.

- Closure: the product of two elements of $L$ also lies in $L$, as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ e & f & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a+d & 1 & 0 \\ b+cd+e & c+f & 1 \end{bmatrix} \in L.$$

#2b). There are lots of choices here. The easiest way to prove you have three nonisomorphic ones is to use cardinality. Note that if you take cyclic subgroups defined in terms of a generator, you know automatically that they are abelian subgroups.

Examples:

$$\left\langle \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle$$ order 2, cyclic

$$\left\langle \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$ order 2, cyclic

$$\left\langle \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle$$ order 3, cyclic

$$\left\langle \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right\rangle$$ countable, cyclic

$$\left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{R}^* \right\}$$ uncountable, abelian, not cyclic.

etc.

#3a) We know from a book proof that $\phi(e_G) = e_H$. (Note they said it was a proof about isomorphisms, but it really only used the homomorphism property, not 1-1 or onto, so it's valid for all homomorphisms.)

Proof: Suppose $a, b$ are inverses in $G$, i.e., $ab = ba = e_G$. Applyin $\phi$ to these equal elements, we get equal results, i.e. $\phi(ab) = \phi(ba) = \phi(e_G)$. By the HP and the result about identifies, we have $\phi(a)\phi(b) = \phi(b)\phi(a) = e_H$, so $\phi(a)$ and $\phi(b)$ are inverses in $H$.

Thus, if $x$ is its own inverse in $G$, we also have that $\phi(x)$ is its own inverse in $H$ (Taking the special case $a = b = x$.)

#3b) Let $g_1, g_2 \in G$. Since $G$ is abelian, we have $g_1 g_2 = g_2 g_1$. Applying $\phi$: $\phi(g_1 g_2) = \phi(g_2 g_1)$. Then by the HP, $\phi(g_1)\phi(g_2) = \phi(g_2)\phi(g_1)$, so elements in the image of $\phi$ commute with each other.

This does NOT imply $H$ is abelian.

Note, to show $H$ really could be non abelian, we need an actual counterexample - it's not enough to point out our current proof is insufficient — maybe we just didn't think of the best proof.

Counter example: Let $G = \{ I_3 \}$, a trivial group consisting of the 3x3 id matrix, and let $H = GL(3, \mathbb{R})$. You can check the inclusion map

$$\varphi : G \to H$$
$$x \mapsto x$$

(Silly as $x = I_3$ is the only element.)

is a homomorphism, but $H$ is not abelian.