113 - HW 2 Solutions

Note: Your solutions should be neater & include problem statements. These solutions are for reference & to help you pick up any parts you missed. Email me ASAP if you find errors. My version also has more commentary. —KT

#1a) Let's check $\sim$ is reflexive, symmetric, & transitive:

- Refl. (WTS: $\langle x,y,z\rangle \sim \langle x,y,z\rangle\ \forall \langle x,y,z\rangle \in V$.)

  Let $\langle x,y,z\rangle \in V$. $x-x = y-y+0 = z-z+2\cdot 0$, so

  $\langle x,y,z\rangle \sim \langle x,y,z\rangle \quad \forall \langle x,y,z\rangle$.

- Symm. (WTS: $\langle x_1,y_1,z_1\rangle \sim \langle x_2,y_2,z_2\rangle \Rightarrow \langle x_2,y_2,z_2\rangle \sim \langle x_1,y_1,z_1\rangle$).

  Let $\langle x_1,y_1,z_1\rangle \sim \langle x_2,y_2,z_2\rangle$; ie

  $x_1-x_2 = y_1-y_2+k = z_1-z_2+2k$ fs $k\in\mathbb{R}$.

  Multiplying all parts by -1, we get $x_2-x_1 = y_2-y_1+(-k)$
  $= z_2-z_1+2(-k)$.

Since $k \in \mathbb{R}$, so is $-k$; thus
$\langle x_2, y_2, z_2 \rangle \sim \langle x_1, y_1, z_1 \rangle$ as desired.

- Transitivity: (WTS: $\langle x_1, y_1, z_1 \rangle \sim \langle x_2, y_2, z_2 \rangle$
and $\langle x_2, y_2, z_2 \rangle \sim \langle x_3, y_3, z_3 \rangle$
$\Rightarrow \langle x_1, y_1, z_1 \rangle \sim \langle x_3, y_3, z_3 \rangle$.)

Let $\langle x_1, y_1, z_1 \rangle \sim \langle x_2, y_2, z_2 \rangle$
and $\langle x_2, y_2, z_2 \rangle \sim \langle x_3, y_3, z_3 \rangle$, i.e.

$$x_1 - x_2 = y_1 - y_2 + k = z_1 - z_2 + 2k$$

and $$x_2 - x_3 = y_2 - y_3 + l = z_2 - z_3 + 2l$$ fs $k, l \in \mathbb{R}$

Adding these two sets of equations, we get

$$x_1 - x_3 = y_1 - y_3 + (k+l) = z_1 - z_3 + 2(k+l),$$

and $k, l \in \mathbb{R} \Rightarrow k+l \in \mathbb{R}$, so

$\langle x_1, y_1, z_1 \rangle \sim \langle x_3, y_3, z_3 \rangle$, which is what we needed to show.

#1b) Description of an equivalence class: ③

Consider an arbitrary $\langle a, b, c\rangle \in V$. Then by the definition of $\sim$, we see

$$\overline{\langle a,b,c\rangle} = \left\{ \langle x, y, z\rangle : \begin{array}{l} x-a = y-b+k = z-c+2k \\ \text{fs } k \in \mathbb{R} \end{array} \right\}.$$

①

This is complete and correct, but it's a little hard to see what geometric object $\overline{\langle a, b, c\rangle}$ is, so we may try rewriting it, with fewer variables, such as

$$\overline{\langle a, b, c\rangle} = \left\{ \langle x, x-a+b-k, x-a+c-2k\rangle : x, k \in \mathbb{R} \right\}.$$

②

If you experiment a bit, you should be able to convince yourself $\overline{\langle a, b, c\rangle}$ is a 2-dimensional object, but it's still not completely clear what. Given the linear nature of the equations (or perhaps wishful thinking), you might suspect you have a plane, and this is correct. The easiest way to verify it is probably to eliminate

k from our system of equations:

$$y-b+k = z-c+2k \implies k = y-b-(z-c).$$

Then $x-a = y-b+k \implies x-a = y-b+\underbrace{y-b-(z-c)}_{k}$,

and we have $x-2y+z = a-2b+c$.

Since a, b, c are constants, you should ~~rcg~~ recognize this as the equation of a plane.

3 So $\langle a,b,c\rangle = \{\langle x,y,z\rangle : x-2y+z = a-2b+c\}$

Note that no matter what $\langle a,b,c\rangle$ is, the normal vector to the plane is $\langle 1,-2,1\rangle$, so these planes are all parallel (which is the only thing that makes sense, since they need to partition V).

$\boxed{\text{Complete list of classes:}}$

~~Either the~~ The planes are angled such that the x-axis will hit each of them in exactly one point.

We can use this fact to give a nice list of all our classes.

<u>Claim</u>: $\{\overline{\langle A, 0, 0\rangle} : A \in \mathbb{R}\}$ is a complete list of equivalence classes with no repetitions. (I.e. each of our planes appears exactly once in this list.)

To check rigorously, we need to show

$$\overline{\langle A_1, 0, 0\rangle} = \overline{\langle A_2, 0, 0\rangle} \Rightarrow A_1 = A_2$$

and that each $\langle a, b, c\rangle$ is in one of these classes.

- Say $\overline{\langle A_1, 0, 0\rangle} = \overline{\langle A_2, 0, 0\rangle}$, i.e.

  $\langle A_1, 0, 0\rangle \sim \langle A_2, 0, 0\rangle$, i.e.

  $A_1 - A_2 = 0 - 0 + k = 0 - 0 + 2k$ fs $k \in \mathbb{R}$.

  $k = 2k \Rightarrow k = 0$, so we must have

  $A_1 = A_2$, and the classes in this

list are all different.

- Now, say $\langle a, b, c\rangle \in V$ is arbitrary.

We need to find an $\langle A, 0, 0\rangle$ class it belongs to, ie some A (in terms of a,b,c) so that $\langle a, b, c\rangle \sim \langle A, 0, 0\rangle$.

We need $a - A = b - 0 + k = c - 0 + 2k$
fs $k \in \mathbb{R}$.

Using the latter two terms, we see $k = b - c$, and then using the first two (or 1^{st} & 3^{rd}),

$$A = a - b - k = a - b - (b - c) = a - 2b + c.$$

So $\langle a, b, c\rangle \sim \langle a - 2b + c, 0, 0\rangle$, and our list of classes covers all of V.

<u>Note</u>: It would also have been fine to use some geometry to explain why we had a complete list of classes.

#2 a) $\mathbb{Z}_6$

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	5	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{5}$	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$

Ok to leave off bars.

U_6

$\cdot$	1	$e^{\frac{\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	-1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{5\pi i}{3}}$
1	1	$e^{\pi i/3}$	$e^{2\pi i/3}$	-1	$e^{4\pi i/3}$	$e^{5\pi i/3}$
$e^{\frac{\pi i}{3}}$	$e^{\pi i/3}$	$e^{2\pi i/3}$	-1	$e^{4\pi i/3}$	$e^{5\pi i/3}$	1
$e^{\frac{2\pi i}{3}}$	$e^{2\pi i/3}$	-1	$e^{4\pi i/3}$	$e^{5\pi i/3}$	1	$e^{\pi i/3}$
-1	-1	$e^{4\pi i/3}$	$e^{5\pi i/3}$	1	$e^{\pi i/3}$	$e^{2\pi i/3}$
$e^{\frac{4\pi i}{3}}$	$e^{4\pi i/3}$	$e^{5\pi i/3}$	1	$e^{\pi i/3}$	$e^{2\pi i/3}$	-1
$e^{\frac{5\pi i}{3}}$	$e^{5\pi i/3}$	1	$e^{\pi i/3}$	$e^{2\pi i/3}$	-1	$e^{4\pi i/3}$

Ok to use

$-1 = e^{\frac{3\pi i}{3}}$

$1 = e^{\frac{0\pi i}{3}}$

Notice the same nice pattern of antidiagonals

#2b) ~~Th~~ The map f is pretty clearly (8)

onto, since the six roots of unity in U_6 are given by $\left(e^{\frac{2\pi i}{6}}\right)^k$ for $k = 0, 1, \ldots, 5$, and $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \ldots, \bar{5}\}$. Since $|\mathbb{Z}_6| = |U_6| = 6$, f is also 1-1.

Let's check it's a homomorphism.

Let $\bar{a}, \bar{b} \in \mathbb{Z}_6$. Then

$f(\bar{a} + \bar{b}) = \left(e^{\frac{2\pi i}{6}}\right)^{a+b}$ and

$f(\bar{a})\,f(\bar{b}) = \left(e^{\frac{2\pi i}{6}}\right)^{a} \cdot \left(e^{\frac{2\pi i}{6}}\right)^{b}$, but these are equal by properties of exponents.

If you are being super careful, you might worry about what happens if $a+b > 6$, so $\bar{a} + \bar{b} = \overline{a+b}$ is not our classic name for the class, ie ~~6-(a+b)~~ $\overline{(a+b)-6}$, but ~~th~~ turns out fine since $\left(e^{\frac{2\pi i}{6}}\right)^6 = 1$.

#2c) Only one map works:

$$g: \mathbb{Z}_6 \to U_6$$

$$\bar{k} \longmapsto \left(e^{\frac{2\pi i}{6}}\right)^{-k} = \left(e^{\frac{2\pi i}{6}}\right)^{6-k}$$

either version; they are equal.

Proof is very similar to 2b).

#3) Only parts b) & c) are shown (& graded). Other 2 are similar.

#3b) Claim: $*$ defined by

$$x * y = x + y + 3 \quad \text{works.}$$

Let's check the homomorphism property.

Let $a, b \in \mathbb{Z}$.

$$\phi : \langle \mathbb{Z}, + \rangle \longrightarrow \langle \mathbb{Z}, * \rangle.$$

$$n \longmapsto n - 3$$

~~$\phi(a+b$~~

$$\phi(a+b) = a + b - 3$$

$$\begin{aligned}\phi(a) * \phi(b) &= (a-3) * (b-3) \\ &= (a-3) + (b-3) + 3 \\ &= a + b - 3.\end{aligned}$$

$\phi(a+b) = \phi(a) * \phi(b)$, so HP holds.

#3c) Claim: $*$ defined by

$$x * y = (x-3)(y-3)+3$$

$$\left(\text{or } = xy - 3x - 3y + 12\right) \text{ works.}$$

Let's check the HP: Let $a, b \in \mathbb{Z}$,

$$\phi : \langle \mathbb{Z}, * \rangle \longrightarrow \langle \mathbb{Z}, \cdot \rangle$$

$$n \longmapsto n-3.$$

$$\begin{aligned}\phi(a * b) &= a*b-3 \\ &= \left[(a-3)(b-3)+3\right]-3 \\ &= (a-3)(b-3)\end{aligned}$$

$$\phi(a)\cdot\phi(b) = (a-3)\cdot(b-3), \text{ so}$$

$\phi(a*b) = \phi(a)\,\phi(b)$, and the HP holds.