MATH 113 MIDTERM 2
Thursday, March 19, 2015

This exam has 5 problems on 9 pages, including this cover sheet. There is also a blank page at the end you may tear off to use as scratch paper. The only thing you may have out during the exam is the exam itself and one or more writing utensils. You have 80 minutes to complete the exam.

DIRECTIONS

- This exam may not be discussed with anyone at all until after 2:00pm on Thursday.
- Be sure to carefully read the directions for each problem. You may work on the problems in any order.
- All work must be done on this exam. If you need more space for any problem, feel free to use the space on the very last page. Draw an arrow or write a note indicating this, so I know where to look for the rest of your work.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- Good luck; do the best you can!

Problem	Max	Score
1	20	
2	10	
3	10	
4	20	
5	40	
Total	100	

1. This problem deals with the first isomorphism theorem for groups.

   (a) (10 points) Prove the first isomorphism theorem for groups, i.e., if $\varphi : G \to H$ is a group homomorphism, then $G/\ker\varphi \cong \operatorname{im}\varphi$, by giving an explicit isomorphism (and showing that is in fact an isomorphism). Be sure to show that your map is well-defined.

Now let's see our theorem in action. Consider the map $\varphi : S_7 \times \mathbb{Z}_{16} \to \mathbb{Z}_2 \times \mathbb{Z}_{24}$ given by $(\sigma, \overline{x}) \mapsto (\operatorname{sign}(\sigma), \overline{3x})$, where $\operatorname{sign}(\sigma)$ is $\overline{0}$ if σ is an even permutation and $\overline{1}$ if σ is an odd permutation.

(b) (5 points) Show that φ is a homomorphism (you may assume it is well-defined), and find its kernel.

(c) (5 points) Determine the isomorphism type of $G/\ker\varphi$ in FTFGAG form by using the first isomorphism theorem. (I.e. do NOT work directly with computing orders of elements in $G/\ker\varphi$. Instead, determine the isomorphism type of $\operatorname{im}\varphi$, which is much easier.)

2. Let $G = \mathbb{Z}_8 \times \mathbb{Z}_{32}$, and let $H = \langle (\overline{2}, \overline{4}) \rangle$.

   (a) (5 points) Prove that G/H is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$.

   (b) (5 points) Prove that G/H is not isomorphic to $\mathbb{Z}_{32}$.

3. (10 points) Prove or disprove the following statement. Be sure it is clear which you are doing.

   If D and E are integral domains, then the direct product $D \times E$ is an integral domain. (You may assume that direct products of rings are rings, using componentwise addition and multiplication.)

4. This problem consists of several short answer questions (not related to each other). You do not need to give explanations in full sentences, but if you have done any computations, your work should show these. Box your answers if they are hard to find in all your work.

   (a) (4 points) Find a subgroup of order 150 in $\mathbb{Z}_{35} \times \mathbb{Z}_{27} \times \mathbb{Z}_{20}$.

   (b) (4 points) Is $\mathbb{Z}_{35} \times \mathbb{Z}_{27} \times \mathbb{Z}_{20}$ isomorphic to $\mathbb{Z}_{300} \times \mathbb{Z}_{63}$? Put both into FTFGAG form to justify your answer.

   (c) (4 points) Give an example of a nontrivial group homomorphism $\varphi : \mathbb{Z}_{72} \to \mathbb{Z}_{42}$.

Short answer continued. You do not need to give explanations in full sentences, but if you have done any computations, your work should show these.

(d) (4 points) Give an example of a group G and a normal subgroup $H \trianglelefteq G$ so that $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(e) (4 points) Give an example of a commutative ring R which has characteristic 7 and is not an integral domain.

5. (2 points each) This problem consists of TRUE/FALSE questions on a variety of topics (there are two pages of these). Remember to pick TRUE only if the statement is ALWAYS true. You will receive 0 points for a wrong answer, 1 point for leaving it blank, and 2 points for a correct answer. No justification is required, though you may use any blank space for scratch work.

   (Group stuff on this page.)

   (a) **TRUE FALSE**
   If $G = D_6$ and $H = \{e, r^2, r^4\}$, then $H \trianglelefteq G$ and $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

   (b) **TRUE FALSE**
   If $K \trianglelefteq H$ and $H \trianglelefteq G$, then $K \trianglelefteq G$.

   (c) **TRUE FALSE**
   There exist groups G and H and a group homomorphism $\varphi : G \to H$ such that $|G| = 72$, $|H| = 42$, and φ is surjective (onto).

   (d) **TRUE FALSE**
   There exist groups G and H and a group homomorphism $\varphi : G \to H$ such that $|G| = 72$, $|H| = 42$, and φ is injective (1-1).

   (e) **TRUE FALSE**
   There exists a cyclic subgroup of order 12 in $S_4 \times \mathbb{Z}_{33}$.

   (f) **TRUE FALSE**
   The kernel of the group homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_{25} \times \mathbb{Z}_3$ determined by $\varphi(1) = (5, 1)$ is $15\mathbb{Z}$.

   (g) **TRUE FALSE**
   The group $\mathbb{Z}_{125} \times \mathbb{Z}_8$ has a subgroup which is isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_5$.

   (h) **TRUE FALSE**
   The group $\mathbb{R}/\mathbb{Z}$ is cyclic.

   (i) **TRUE FALSE**
   All elements of the group $\mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_4$ have infinite order.

   (j) **TRUE FALSE**
   There are at least 10 nonisomorphic groups of order $2^3 \cdot 3^4$.

(Ring stuff on this page.)

(k) **TRUE FALSE**
The polynomial ring $\mathbb{R}[x]$ has infinitely many subrings.

(l) **TRUE FALSE**
The lower triangular matrices of $M_3(\mathbb{C})$ form a subring of $M_3(\mathbb{C})$.

(m) **TRUE FALSE**
The map $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_{36}$ given by $\overline{x} \mapsto \overline{3x}$ is a ring homomorphism.

(n) **TRUE FALSE**
A finite integral domain must be a field.

(o) **TRUE FALSE**
The ring $M_2(\mathbb{R})$ has infinitely many zero divisors.

(p) **TRUE FALSE**
If D is an integral domain, then the matrix ring $M_3(D)$ with entries from D is also an integral domain.

(q) **TRUE FALSE**
If D is an integral domain, then the polynomial ring $D[x]$ with coefficients in D is also an integral domain.

(r) **TRUE FALSE**
If F_1 and F_2 are fields, then $F_1 \times F_2$ is a field.

(s) **TRUE FALSE**
The set $\{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{C}$.

(t) **TRUE FALSE**
If $\operatorname{char} R = k$ and $\operatorname{char} S = \ell$, where k and ℓ are both positive integers, then $\operatorname{char}(R \times S) = lcm(k, \ell)$.