# MATH 113 MIDTERM
## Thursday, February 19, 2015

This exam has 5 problems on 9 pages, including this cover sheet. There is also a blank page at the end you may tear off to use as scratch paper. The only thing you may have out during the exam is the exam itself and one or more writing utensils. You have 80 minutes to complete the exam.

### DIRECTIONS

- This exam may not be discussed with anyone at all until after 2:00pm on Thursday.
- Be sure to carefully read the directions for each problem.
- All work must be done on this exam. If you need more space for any problem, feel free to use the space on the very last page. Draw an arrow or write a note indicating this, so I know where to look for the rest of your work.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- Good luck; do the best you can!

| Problem | Max | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 20 | |
| 5 | 40 | |
| Total | 100 | |

1. In this problem, we will (mostly) prove Lagrange's Theorem. Please note it continues onto the next page.

   (a) (4 points) Carefully state Lagrange's Theorem. Use $G$ to denote your main group, and $H$ to denote a subgroup of $G$.

   (b) (6 points) The first step of the proof partitions $G$ into left cosets using an equivalence relation. Prove that the relation below is an equivalence relation on $G$. Give complete details.

$$a \sim b \text{ if and only if } a^{-1}b \in H$$

(*Lagrange proof, continued.*)

(c) (5 points) The next step of the proof checks that all cosets have the same size. Summarize the proof of this in **1-3 sentences**; do not give complete details. If a map or relation is involved, be sure to describe it. (If you run out of space, you are probably saying too much.)

(d) (5 points) Complete the proof of Lagrange's Theorem using parts (b) and (c).

2. (10 points) Let $\phi : G \to H$ be a group homomorphism, and let $B$ be a subgroup of $H$. Let $A = \phi^{-1}[B]$, i.e. $A = \{g \in G : \phi(g) \in B\}$. Prove that $A$ is a subgroup of $G$.

3. (10 points) Prove or disprove the following statement. Be sure it is clear which you are doing.

   Let $G = D_5$, and let $H = \{\sigma \in D_5 : \sigma^2 = e\}$. Then $H$ is a subgroup of $D_5$.

4. This problem consists of several short answer questions (not related to each other). You do not need to give explanations in full sentences, but if you have done any computations, you work should show these. (Even easy computations like $lcm(6,8)$.)

   (a) (5 points) Find the order of the permutation $\sigma = (1,2,4,5)(2,6)(1,6,3)(1,3,7,9)$ in $S_9$. Is $\sigma \in A_9$?

   (b) (5 points) Let $G$ be a cyclic group of order 75. How many subgroups of order 15 does $G$ have? How many elements of order 15?

*Short answer continued. You do not need to give explanations in full sentences, but if you have done any computations, you work should show these.*

(c) (5 points) In homework, we showed that $H = \{(x, y, z) \in \mathbb{R}^3 : 3x + 2y - 2z = 0\}$ is a subgroup of $G = \mathbb{R}^3$. What geometric object is the coset $(1, 2, 3)H$ in $G$? What is the index of $H$ in $G$?

(d) (5 points) What is the smallest subgroup of $S_5$ that contains $(1, 2)$ and $(2, 4)$? What familiar group is it isomorphic to?

5. (2 points each) This problem consists of TRUE/FALSE questions on a variety of topics (there are two pages of these). Remember to pick TRUE only if the statement is ALWAYS true. You will receive 0 points for a wrong answer, 1 point for leaving it blank, and 2 points for a correct answer. No justification is required, though you may use any blank space for scratch work.

   (a) **TRUE FALSE**
   The group $D_{12}$ has a subgroup isomorphic to $D_4$.

   (b) **TRUE FALSE**
   Every nontrivial subgroup of $GL(4, \mathbb{R})$ is infinite.

   (c) **TRUE FALSE**
   Every proper subgroup of $GL(2, \mathbb{R})$ is cyclic.

   (d) **TRUE FALSE**
   The symmetry group of a circle is finite.

   (e) **TRUE FALSE**
   If $K$ is a subgroup of $H$ and $H$ is a subgroup of $G$, then $K$ is a subgroup of $G$.

   (f) **TRUE FALSE**
   If $H$ is a subgroup of $S_3$, then the partition of $S_3$ into left cosets of $H$ is the same as the partition of $S_3$ into right cosets of $H$.

   (g) **TRUE FALSE**
   Every subgroup of $U_{24}$ is cyclic.

   (h) **TRUE FALSE**
   The group $D_{48}$ has a subgroup of order $k$ for every positive factor $k$ of 48.

   (i) **TRUE FALSE**
   If $G$ is an abelian group of order $n$, then $G$ is isomorphic to $\mathbb{Z}_n$.

   (j) **TRUE FALSE**
   The element $r^2sr^5srs$ in $D_9$ has order 2.

(k) **TRUE FALSE**
A group of prime order can have a proper nontrivial subgroup.

(l) **TRUE FALSE**
If a group $G$ has order 13, then $G$ must be abelian.

(m) **TRUE FALSE**
The set of functions $\{f : \mathbb{R} \to \mathbb{R}\}$ is a group under (pointwise) function multiplication, and its identity element is the function $f_I : \mathbb{R} \to \mathbb{R}$ defined by $f_I(x) = 1$ for all $x \in \mathbb{R}$.

(n) **TRUE FALSE**
Lagrange's Theorem shows that $\mathbb{Z}$ has a subgroup of order $k$ for every positive integer $k$.

(o) **TRUE FALSE**
If $\phi : \mathbb{Z}_{10} \to U_{10}$ is a group homomorphism, then $\phi(\overline{0}) = 1$.

(p) **TRUE FALSE**
Every odd permutation can be written as a product of three transpositions.

(q) **TRUE FALSE**
There exists a nonabelian group of order 22 whose proper subgroups are all abelian.

(r) **TRUE FALSE**
If $H$ is a finite subgroup of $\mathbb{R}$, then there are infinitely many right cosets of $H$ in $\mathbb{R}$.

(s) **TRUE FALSE**
The group $S_{15}$ has a subgroup $K$ which is not a subgroup of $A_{15}$.

(t) **TRUE FALSE**
If $G$ is a group, then the map $\phi : G \to G$ defined by $x \mapsto x^{-1}$ for all $x \in G$ is a group homomorphism.