MATH 113 FINAL EXAM
May 13-14, 2015

This exam has 8 problems on 17 pages, including this cover sheet. The only thing you may have out during the exam is one or more writing utensils. You have 180 minutes to complete the exam.

DIRECTIONS

- Be sure to carefully read the directions for each problem.
- All work must be done on this exam. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this, so I know where to look for the rest of your work.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- Good luck – do the best you can!

Problem	Max	Score
1	40	
2	10	
3	45	
4	55	
5	15	
6	15	
7	15	
8	10	
Total	205	

1. The parts of this problem are not related to each other. Your justifications should be very brief, and you don't need to use complete sentences.

   (a) (5 points) Find a polynomial $f(x)$ with five terms that is irreducible over $\mathbb{Q}$ by Eisenstein's Criterion. Let α be a root of $f(x)$. Give a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. (You don't need to figure out what real/complex number α is.)

   (b) (5 points) Assuming β is transcendental over $\mathbb{Q}$, show that β^7 is transcendental over $\mathbb{Q}$.

(c) (5 points) Consider the congruence $55x \equiv 115 \pmod{75}$. Find all solutions in $\mathbb{Z}_{75}$, showing your work.

(d) (5 points) Find a pair of zero divisors in the ring $M_2(\mathbb{Z}_8)$ (consisting of 2×2 matrices with entries from $\mathbb{Z}_8$).

(e) (5 points) In the factor group $\mathbb{Z}_8 \times \mathbb{Z}_4/\langle(2,2)\rangle$, find the order of the coset $(5,1) + \langle(2,2)\rangle$.

(f) (5 points) Find the order of the element $(1,4,7,2,8,9)(2,3,4,5)(3,8,5)$ in the group S_{10}.

(g) (5 points) Show that the quotient group $\mathbb{Z}_9 \times \mathbb{Z}_2 / \langle (6, 0) \rangle$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_2$ by finding an appropriate group homomorphism and applying the first isomorphism theorem. (You do not need to rigorously show that you have a homomorphism.)

(h) (5 points) Suppose $\phi : G \to H$ is a group homomorphism which is NOT one-to-one. Suppose $|G| = 36$ and G has NO normal subgroups of order 2, 3, 4, or 6. List all groups that $im(\phi)$ could possibly be isomorphic to.

2. (10 points) Construct a field with 8 elements, by taking an appropriate quotient of a polynomial ring. Be sure to describe the elements of your field and justify your work, quoting any relevant theorems you need.

3. (3 points each) No justification is required, but you may use the space to do (ungraded) scratch work if you want. Circle the correct answer, and make sure there is no ambiguity if you change your mind. Note: normal TF, no partial credit for blanks.

   (a) If G is a group and H and K are two subgroups of G, then $H \cap K$ and $H \cup K$ are both subgroups of G.

   TRUE *FALSE*

   (b) The groups $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{42}$ and $\mathbb{Z}_{36} \times \mathbb{Z}_{28} \times \mathbb{Z}_3$ are isomorphic.

   TRUE *FALSE*

   (c) Every factor group of a cyclic group is cyclic.

   TRUE *FALSE*

   (d) If g_1 and g_2 are elements of a group G, then $|g_1 g_2| = |g_1| \cdot |g_2|$.

   TRUE *FALSE*

   (e) In the group $\mathbb{Q}/\mathbb{Z}$ under addition, every element has finite order.

   TRUE *FALSE*

(f) There exists an integral domain of characteristic 15.

TRUE *FALSE*

(g) In the ring $\mathbb{Z}[x]$, the polynomials of the form $a_0 + a_1x^5 + a_2x^{10} + a_3x^{15} + \cdots a_nx^{5n}$ with $a_i \in \mathbb{Z}$ form an ideal.

TRUE *FALSE*

(h) In a commutative ring, every prime ideal must also be maximal.

TRUE *FALSE*

(i) There exists a commutative ring R with a noncommutative factor ring R/I.

TRUE *FALSE*

(j) The direct product of two noncommutative rings is a noncommutative ring.

TRUE *FALSE*

(k) The element $\sqrt{5}+\sqrt{7}$ is algebraic of degree 4 over $\mathbb{Q}$.

TRUE *FALSE*

(l) If E is an extension field of F, and $[E:F]$ is prime, then there must exist $\alpha \in E$ such that $E = F(\alpha)$.

TRUE *FALSE*

(m) The field $\mathbb{Q}(i)$ is algebraically closed.

TRUE *FALSE*

(n) The field $\mathbb{Q}(\sqrt[5]{13})$ is a dimension 5 vector space over $\mathbb{Q}$.

TRUE *FALSE*

(o) If $F(\alpha)$ is a simple algebraic extension of a field F, and $\beta \in F(\alpha)$, then β is a root of a polynomial in $F[x]$.

TRUE *FALSE*

4. (5 points each) For each of the items listed below, give a *specific* example with the stated property. All of these are possible, and no justification is required.

   (a) An ideal of $\mathbb{Z}_5 \times \mathbb{Z}_4$ which is not a prime ideal.

   (b) A finite field with at least 20 elements.

   (c) An integer k such that $\varphi(k) = 8$, where φ denotes Euler's function.

(d) A polynomial ring which is not an integral domain.

(e) A commutative subring of the noncommutative ring $M_n(\mathbb{Q})$.

(f) An integral domain R and and ideal I such that R/I has zero divisors.

(g) A nontrivial ring homomorphism $\mathbb{Z}[x] \to \mathbb{Z} \times \mathbb{Z}$.

(h) Three rings R, S, and T with different characteristics, such that the direct product ring $R \times S \times T$ has characteristic 20.

(i) A field F which contains $\mathbb{Q}$ as a proper subfield and is a proper subfield of $\mathbb{Q}(\sqrt[6]{13})$, i.e. $\mathbb{Q} \lneq F \lneq \mathbb{Q}(\sqrt[6]{(13)})$.

(j) A nonabelian group with at least ten elements of order 7.

(k) A subgroup of $D_4 \times S_3$ which has 4 left cosets.

5. (15 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) Let H be a normal subgroup of G of index m. Prove that $g^m \in H$ for all $g \in G$. (Hint: use what you know about G/H.)

   (b) Let $\phi : G \to H$ be a group homomorphism between two finite groups. Prove that the size of the image, i.e. $|im(\phi)|$, is a divisor of both $|G|$ and $|H|$.

6. (15 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) Let R be a commutative ring, and let I be an ideal of R. Prove that I is a prime ideal if and only if the factor ring R/I is an integral domain.

   (b) Let $\phi : R \to S$ be a ring homomorphism. If I' is an ideal of S, prove that the inverse image $I = \phi^{-1}(I')$ is an ideal of R.

7. (15 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   These are both portions of the proof of Kronecker's Theorem. Let $p(x)$ be an irreducible polynomial of degree at least 2 in the polynomial ring $F[x]$, where F is a field. Let I be the ideal $I = \langle p(x) \rangle$, and consider the factor ring $F[x]/I$.

   (a) Show that the elements of $F[x]/I$ are in bijection with the possible remainders after dividing by $p(x)$ (as described in the Division Algorithm).

   (b) If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$, view it as a polynomial whose inputs are cosets using

   $$p(\text{coset}) = (a_0 + I) + (a_1 + I)(\text{coset})^1 + (a_2 + I)(\text{coset})^2 + \cdots + (a_n + I)(\text{coset})^n$$

   Prove that the coset $x + I$ is a root of $p(x)$. Be sure you justify all steps.

8. (a) (5 points) What is your favorite group and why?

(b) (5 points) What is your favorite 113 theorem? Briefly describe (3-5 sentences) something that you like about the proof or about an application of the theorem you choose.