Start 11:20

end

MATH 113 MIDTERM
Thursday, March 7, 2013

This exam has 6 problems on 9 pages, including this cover sheet. The only thing you may have out during the exam is one or more writing utensils. You have 80 minutes to complete the exam.

DIRECTIONS

- Be sure to carefully read the directions for each problem.
- All work must be done on this exam. If you need more space for any problem, feel free to continue your work on the back of the page or on the blank page at the end of the test. Draw an arrow or write a note indicating this, so I know where to look for the rest of your work.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- Good luck; do the best you can!

Problem	Max	Score
1	20	
2	10	
3	30	
4	10	
5	10	
6	20	
Total	100	

1. For all parts of this problem, $G = \mathbb{Z}_6 \times \mathbb{Z}_8$ and H is the subgroup $\langle(4,4)\rangle$ of G.

   (a) (5 points) What is the order of the group G/H?

   $|H| = \text{lcm}(3, 2) = 6$

   $|G/H| = |G|/|H| = \frac{48}{6} = \boxed{8}$

   (b) (5 points) Give a complete list of all isomorphism types of abelian groups of order $|G/H|$. Write each group/type in FTFGAG form.

   $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

   $\mathbb{Z}_2 \times \mathbb{Z}_4$

   $\mathbb{Z}_8$

   (c) (5 points) Consider the coset $(3,1)+H$. What is its order as an element of G/H?

   $H = \{(0,0), (4,4), (2,0), (0,4), (4,0), (2,4)\}$.

   $(3,1) \cdot 2 = (0,2)$

   $(3,1) \cdot 3 = (3,3)$

   $(3,1) \cdot 4 = (0,4) \in H$

   $\boxed{\text{order } 4}$

   (d) (5 points) Using your answer from part (c), can you show that G/H is *NOT* isomorphic to any of the groups you listed in part (b)? Briefly justify your answer. (Do not compute the orders of any more cosets; base your answer only on information we have so far.)

   $G/H \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ since all elts here have order ≤ 2.

2. (2 points each) Circle the correct answer. No justification is required.

   (a) If a group G has a subgroup of order 4, then G must have an even number of elements.

   TRUE *FALSE*

   (b) If G is a cyclic group, then every factor group of G is also cyclic.

   TRUE *FALSE*

   (c) If $\phi : G \to H$ is a group homomorphism, $|G| = 12$, and $|H| = 42$, then the largest possible size of the image $\phi[G]$ is 12.

   TRUE *FALSE*

   (d) The group $\mathbb{Z}_2 \times \overset{2}{\mathbb{Z}_6} \times \overset{2}{\mathbb{Z}_{14}} \times \overset{5}{\mathbb{Z}_{35}}$ has a subgroup of order 20.

   TRUE *FALSE*

   (e) There are at least five nonisomorphic abelian groups of order 72.

   TRUE *FALSE*

$72 = 2^3 \cdot 3^2$

3 ways, 2 ways = 6 gps

3. (5 points each) For each of the items listed below, give a SPECIFIC example with the stated property. Do not simply say why an example exists. All of these are possible.

   (a) A cyclic subgroup of order 12 in S_7.

   $$\langle (1,2,3)(4,5,6,7) \rangle$$

   (b) A nonabelian group of order 14 whose proper subgroups are all abelian.

   D_7

   (c) A group G and a proper nontrivial subgroup H such that there are 5 cosets of H in G.

   $G = \mathbb{Z}$

   $H = 5\mathbb{Z}$.

(d) A group of order 24 which has more than one subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

(e) A group G and a homomorphism $\phi : \mathbb{Z} \to G$ whose kernel is $5\mathbb{Z}$.

$$G = \mathbb{Z}_5$$

$$\phi(k) = k \bmod 5$$
$$= \bar{k}$$

(f) A group G with at least ten elements of order 5.

$$\mathbb{Z}_5 \times \mathbb{Z}_5$$

4. (10 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) Let $\phi : G \to G'$ be a group homomorphism. Prove that if g is an element of order k in G, then $\phi(g)$ has order at most k in G'.

   (b) Let $\phi : G \to G'$ be a group homomorphism. Prove that if G is abelian, then $\phi[G]$ is also abelian.

$g^k = e$, so

$$\phi(g^k) = \phi(e) = e'$$

$$\parallel$$

i.e. $(\phi(g))^k = e'$

$$\Rightarrow |\phi(g)| \leq k.$$

b) $\phi : G \xrightarrow[\text{homom}]{} G'$, G abelian

WTS: $\phi[G]$ abelian

Pf: Suppose $a', b' \in \phi[G]$, i.e.

$a' = \phi(a)$, $b' = \phi(b)$ f.s. $a, b \in G$.

Then $a'b' = \phi(a)\phi(b)$

$= \phi(ab)$ homom prop

$= \phi(ba)$ G abelian

$= \phi(b)\phi(a)$ homom prop

$= b'a'$.

Since a', b' are arbitrary elements of $\phi[G]$, we see $\phi[G]$ is abelian.

5. (10 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) Suppose G is a group with two normal subgroups H and K. Prove that $H \cap K$ is a normal subgroup of G. (In particular, you must first show it is a subgroup.)

   (b) Consider the subset $SL(n, \mathbb{R}) = \{M \in GL(n, \mathbb{R}) : \det(M) = 1\}$ of the group $GL(n, \mathbb{R})$. Prove that $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$. (In particular, you must first show it is a subgroup.)

$e \in H, K$, so $e \in H \cap K$

If $x \in H \cap K$, then $x \in H, K$. Since $H, K \leq G$, $x^{-1} \in H, K$, so $x^{-1} \in H \cap K$.

If $x, y \in H \cap K$, then $x, y \in H, K$. So $xy \in H, K$, and $xy \in H \cap K$.

Normal: we know

$gxg^{-1} \in H \quad \forall x \in H, g \in G.$

$gxg^{-1} \in K \quad \forall x \in K, g \in G,$

so $gxg^{-1} \in H \cap K \quad \forall x \in H \cap K, g \in G.$

$\Rightarrow H \trianglelefteq G.$

b) Prove $SL(n,\mathbb{R}) \trianglelefteq GL(n,\mathbb{R})$.

pf: First we show $SL(n,\mathbb{R}) \leq GL(n,\mathbb{R})$ w/ subgroup criterion.

- Clearly I_n = $n \times n$ id. matrix is in $SL(n,\mathbb{R})$.
- Suppose $A, B \in SL(n,\mathbb{R})$. Then

$$\det(AB^{-1}) = \det(A)\det(B)^{-1} = 1 \cdot 1^{-1} = 1$$

so $AB^{-1} \in SL(n,\mathbb{R})$.

By subgroup criterion, $SL(n,\mathbb{R}) \leq GL(n,\mathbb{R})$.

Next, suppose $A \in SL(n,\mathbb{R})$, $M \in GL(n,\mathbb{R})$.

Then $\det(MAM^{-1}) = \det(M)\det(A)\det(M^{-1})$
$= \det(M) \cdot 1 \cdot (\det(M))^{-1}$
$= 1$, so $MAM^{-1} \in SL(n,\mathbb{R})$,

which implies $SL(n,\mathbb{R}) \trianglelefteq GL(n,\mathbb{R})$

6. (5 points each) The following short answer questions are all unrelated.

(a) Find the order of the element $(1, 4, 7, 8)(2, 4, 5)(6, 9)$ in the group S_9.

$= (1, 4, 5, 2, 7, 8)(3)(6, 9)$

length 6 2

$\mathrm{lcm}(6, 2) = 6$.

(b) Let $\phi : \mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}_4$ be the homomorphism defined by $\phi(1) = (\overline{1}, \overline{2})$. Find $\ker(\phi)$.

$\phi(k) = (\bar{k}, \overline{2k})$. this is

$(\bar{0}, \bar{0})$ when $3|k$ and $2|k$,

ie $6|k$. so

$\ker\phi = 6\mathbb{Z}$.

(c) Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_{15} \times \mathbb{Z}_{18}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ isomorphic? Why or why not?

- $\mathbb{Z}_4 \times (\mathbb{Z}_3 \times \mathbb{Z}_5) \times (\mathbb{Z}_2 \times \mathbb{Z}_9)$
- $\mathbb{Z}_3 \times (\mathbb{Z}_4 \times \mathbb{Z}_9) \times (\mathbb{Z}_2 \times \mathbb{Z}_5)$

same list of factors, so

$\cong$ by FTFGAG.

(d) What is your favorite algebra theorem so far?

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