MATH 113 FINAL EXAM
May 17, 2013

This exam has 8 problems on 17 pages, including this cover sheet. The only thing you may have out during the exam is one or more writing utensils. You have 180 minutes to complete the exam.

DIRECTIONS

- Be sure to carefully read the directions for each problem.
- All work must be done on this exam. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this, so I know where to look for the rest of your work.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- Good luck – do the best you can!

Problem	Max	Score
1	40	
2	10	
3	45	
4	55	
5	15	
6	15	
7	15	
8	10	
Total	205	

1. The parts of this problem are not related to each other. Your justifications should be very brief, and you don't need to use complete sentences.

   (a) (5 points) Recall that elements of the dihedral group D_n can all be expressed in the form $r^a s^b$, where r is the small rotation, i.e. rotation by $\frac{360}{n}^\circ$; s is any reflection; $a \in \{0, 1, \ldots, n-1\}$; and $b \in \{0, 1\}$. Show that $D_n/\langle r \rangle \cong \mathbb{Z}_2$ by finding an appropriate group homomorphism and applying the first isomorphism theorem.

   (b) (5 points) Suppose π^5 is algebraic over a field F, and its irreducible polynomial over F is $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Explain why π must also be algebraic over F.

(c) (5 points) Find $\varphi(16)$, where φ denotes Euler's phi-function. Then use Euler's Theorem to find the remainder when 7^{103} is divided by 16. Show all of your calculations in an organized manner.

(d) (5 points) Find all solutions of the equation $39x = 21$ in Z_{27}. Show all relevant work.

(e) (5 points) Suppose $\phi : G \to H$ is a group homomorphism which is NOT one-to-one. Suppose $|G| = 36$ and G has NO normal subgroups of order 2, 3, 4, or 6. List all groups that $im(\phi)$ could possibly be isomorphic to.

(f) (5 points) For which n will the symmetric group S_n have exactly n even elements and exactly n odd elements? Justify your answer.

(g) (5 points) Using the fact that 3 is a zero of the polynomial $p(x) = x^3 - x^2 + x + 1$ in $\mathbb{Z}_{11}$, factor $p(x)$ as a product of a linear factor and a quadratic factor in $\mathbb{Z}_{11}[x]$.

(h) (5 points) Find an integer n and an ideal I of the ring $R = n\mathbb{Z}$ such that the quotient ring R/I has two elements but is not isomorphic to $\mathbb{Z}_2$. Fill in the addition and multiplication tables for R/I below.

$R =$

$I =$

R/I :

$+$		

$\cdot$		

2. (10 points) Construct a field with 49 elements, by taking an appropriate quotient of a polynomial ring. Be sure to describe the elements of your field and justify your work, quoting any relevant theorems you need.

3. (3 points each) No justification is required, but you may use the space to do (ungraded) scratch work if you want. Circle the correct answer, and make sure there is no ambiguity if you change your mind.

   (a) If G is an infinite group, then G must have an element of infinite order.

   $TRUE$ $FALSE$

   (b) The FTFGAG implies that the groups $\mathbb{Z}_4 \times \mathbb{Z}_{15} \times \mathbb{Z}_{18}$ and $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_9 \times \mathbb{Z}_{10}$ are isomorphic.

   $TRUE$ $FALSE$

   (c) If G is a cyclic group, then every subgroup of G and every factor group of G is also cyclic.

   $TRUE$ $FALSE$

   (d) The group S_9 has at least one element of order 60.

   $TRUE$ $FALSE$

   (e) The group S_9 has at least one subgroup of order 60.

   $TRUE$ $FALSE$

(f) The rational root theorem implies that $2x^5 + 7x^3 - x + 1$ is irreducible over $\mathbb{Q}[x]$.

$TRUE$ $FALSE$

(g) If G is an infinite group and $H \trianglelefteq G$, then G/H is an infinite group.

$TRUE$ $FALSE$

(h) In S_5, the subset consisting of the identity plus all cycles of length 2 or 3 is a subgroup of S_5.

$TRUE$ $FALSE$

(i) The matrix group $GL(n, \mathbb{C})$ is nonabelian for all $n \geq 2$.

$TRUE$ $FALSE$

(j) The factor ring $\mathbb{Z}[x]/\langle x^2 + x\rangle$ is an integral domain.

$TRUE$ $FALSE$

(k) If p and q are different primes, then the polynomial $x^2 - 1$ has exactly two zeroes in $\mathbb{Z}_p \times \mathbb{Z}_q$.

TRUE *FALSE*

(l) The ring $\mathbb{Z}_3[x]$ has only finitely many ideals.

TRUE *FALSE*

(m) The direct product of two fields (with at least two elements each) can be a field.

TRUE *FALSE*

(n) A ring with characteristic zero must be infinite.

TRUE *FALSE*

(o) The subset $\mathbb{Q}$ is an ideal of the ring $\mathbb{R}$.

TRUE *FALSE*

4. (5 points each) For each of the items listed below, give a *specific* example with the stated property. All of these are possible, and no justification is required.

   (a) A group G and two elements $x, y \in G$ such that $|x| = 2$, $|y| = 5$, and $|xy| < 10$.

   (b) A subgroup H of $G = S_5 \times \mathbb{Z}_{12}$ such that there are 8 cosets of H in G.

   (c) A (multiplicative) group G whose elements are 2×2 matrices and a proper non-trivial normal subgroup N of G.

(d) A pair of zero divisors in $M_2(\mathbb{Z}_6)$.

(e) A integral domain D and an ideal I such that the factor ring D/I is NOT an integral domain.

(f) A nontrivial ring homomorphism $\phi : \mathbb{Z}[x] \to \mathbb{Z} \times \mathbb{Z}_5$.

(g) A maximal ideal I of the ring $\mathbb{Z}_2 \times \mathbb{Z}_4$.

(h) A factor ring of $R = \mathbb{Z}_6[x]$ which is an integral domain.

(i) A polynomial $f(x)$ in $\mathbb{Z}[x]$ which has degree 25 and can be shown to be irreducible using Eisenstein's Criterion with $p = 5$.

(j) A complex number β which is algebraic of degree 2 over $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

(k) A basis for the field extension $\mathbb{Q}(2 + \sqrt{5}, i)$, viewed as a vector space over $\mathbb{Q}$.

5. (15 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) Suppose E is a finite extension of a field F, and $[E : F]$ is a prime number. Prove that if $\alpha \in E$, then either $F(\alpha) = F$ or $F(\alpha) = E$.

   (b) Suppose E is a finite extension of a field F. Let $p(x) \in F[x]$ be a polynomial which is irreducible over F. Prove that if E contains a zero of $p(x)$, then the degree of $p(x)$ divides $[E : F]$.

6. (15 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) Suppose R is a commutative ring with unity and let I be an ideal of R. Prove that if R/I is an integral domain, then I is a prime ideal of R.

   (b) Let A and B be two ideals of a ring R. Prove that $A+B = \{a+b : a \in A, b \in B\}$ is also an ideal of R.

7. (15 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) State and prove Lagrange's theorem.

   (b) Prove this special case of the first isomorphism theorem for groups, without citing the more general version. Suppose $\phi : G \to H$ is a surjective group homomorphism, and let $N = \ker(\phi)$. Prove that the map $\mu : G/N \to H$ given by $\mu(xN) = \phi(x)$ is a group isomorphism. (Note G is a group written in multiplicative notation.)

8. (a) (5 points) What is your favorite group and why?

(b) (5 points) What is your favorite 113 theorem? Briefly describe (3-5 sentences) something that you like about the proof or about an application of the theorem you choose.