+ (extra for green parts).

MATH 113 PRACTICE MIDTERM
Friday, October 5, 2012

This exam has 5 problems on 8 pages, including this cover sheet and one blank page at the end (which you may use for scratch work, if you desire). The only thing you may have out during the exam is one or more writing utensils. You have 80 minutes to complete the exam.

Real thing will be a tiny bit shorter or easier.

DIRECTIONS

- Be sure to carefully read the directions for each problem.
- All work must be done on this exam. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this, so I know where to look for the rest of your work.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- Good luck; do the best you can!

Problem	Max	Score
1	30	
2	30	
3	10	
4	10	
5	20	
Total	100	

1. (5 points each) For each of the following, the answer is worth 2 points, and the justification is worth 3 points. Circle the correct answer, and give a very brief justification of your answer (quote appropriate theorems, show relevant calculations, give a counterexample, etc.).

   (a) There exist two groups G and H such that $|G| = 12$, $|H| = 8$, and there is a surjective homomorphism $\varphi : G \to H$.

   TRUE *FALSE*

   Thm: $|\varphi[G]|$ must divide $|G|$.

   (b) Suppose g is an element of a group G and $|g| = 10$. Then $|g^2| = 5$.

   TRUE *FALSE*

   $g^{10} = (g^2)^5 = 1$

   $(g^2)^k \neq 1$ if $0<k<5$, since $|g|=10$

   (c) The factor group $\mathbb{R}/\mathbb{Z}$ (under addition) has at least three elements of order 11.

   TRUE *FALSE*

   eg: $\frac{1}{11} + \mathbb{Z}$

   $\frac{2}{11} + \mathbb{Z}$

   $\frac{3}{11} + \mathbb{Z}$

(d) The set of positive rationals is a subgroup of $\mathbb{C}$ under addition.

TRUE *FALSE*

$\mathbb{Q}^+$ doesn't contain inverses of its elements.

(e) Every subgroup of a nonabelian group is nonabelian.

TRUE *FALSE*

$\langle (1,2) \rangle$ in S_3 is abelian, but S_3 is nonabelian

(f) Every nonidentity element of a cyclic group $\mathbb{Z}_n$ generates the whole group.

TRUE *FALSE*

In $\mathbb{Z}_6$, 3 generates a subgroup of order 2, not all of $\mathbb{Z}_6$.

2. (5 points each) For each of the items listed below, give an example with the stated property. All of these are possible.

(a) A finite abelian group with at least five elements of order 3.

$\mathbb{Z}_3 \times \mathbb{Z}_3$

(b) A nontrivial homomorphism from $\mathbb{Z}_{12} \times D_4$ to $\mathbb{Z}_4$.

$\varphi : \mathbb{Z}_{12} \times D_4 \longrightarrow \mathbb{Z}_4$

$(x, y) \longmapsto x \bmod 4$

(c) A subgroup of $S_3 \times Z_4$ which has exactly 8 elements.

$\langle (1,2) \rangle \times \mathbb{Z}_4$

(d) An infinite group G and a subgroup $H \leq G$ such that there are infinitely many left cosets of H in G.

$G = \mathbb{Q}$

$H = \mathbb{Z}$

(e) A subgroup of $GL(2, \mathbb{C})$ which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \begin{array}{l} a \in \{1, -1\} \\ b \in \{1, -1, i, -i\} \end{array} \right\}$$

(f) A subgroup of S_9 which is isomorphic to $\mathbb{Z}_3 \times V$, where V denotes the Klein 4-group.

$\cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\langle (1,2,3), (4,5), (6,7) \rangle$$

3. Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) Suppose G is a group, and H and K are subgroups of G. Prove that the intersection $H \cap K$ is a subgroup of G.

   (b) Suppose $\varphi : G \to G'$ is a group homomorphism. Prove that $\varphi[G]$ is a subgroup of G'.

We need to check 4 things:

- Identity: We know $e \in H$ and $e \in K$, since H and K are subgroups of G, so $e \in H \cap K$.
- Inverses: Suppose $g \in H \cap K$, then $g \in H$ and $g \in K$. Since $g \in H$ and H is a group, $g^{-1} \in H$. Similarly, $g^{-1} \in K$. Thus $g^{-1} \in H \cap K$.
- Closure: Suppose $g_1, g_2 \in H \cap K$. Then $g_1, g_2 \in H$ and since H is a group, $g_1 g_2 \in H$. Similarly, $g_1 g_2 \in K$, so $g_1 g_2 \in H \cap K$.
- Associativity we get for free since everything is in G.

We have showed that $H \cap K$ satisfies all the necessary properties to be a subgroup of G.

b) $\varphi: G \to G'$. Show $\varphi[G] \leq G'$.

We need four things to show $\varphi[G] \leq G'$.

- Assoc. - ok. since G' is a group.
- Identity - We've proved that homomorphisms take e to e' so $e' \in \varphi[G]$.
- Inverses - Suppose g' is in $\varphi[G]$, i.e. $\exists\, g \in G$ such that $\varphi(g) = g'$. We know homomorphisms take inverses to inverses, so $\varphi(g^{-1}) = (g')^{-1}$, and thus $(g')^{-1} \in \varphi[G]$.
- Closure - Suppose $g_1', g_2' \in \varphi[G]$. Then $\exists$ $g_1, g_2 \in G$ such that $\varphi(g_1) = g_1'$, $\varphi(g_2) = g_2'$. Then since φ is a homomorphism,

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) = g_1' g_2', \text{ so}$$

$$g_1' g_2' \in \varphi[G].$$

Thus, $\varphi[G]$ is a subgroup of G'.

4. Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) Suppose that G is a cyclic group and H is a subgroup of G. Prove that G/H is cyclic.

   (b) Let H be a normal subgroup of G of index m. Prove that $g^m \in H$ for all $g \in G$. (Hint: use what you know about G/H.)

Suppose G is cyclic and g is a generator of G. We claim gH generates G/H.

Since $G = \langle g \rangle$, we know all elements of G can be written in the form g^k for some integer $k \geq 0$. Thus, all cosets of H have the form $g^k H$. But $g^k H = (gH)^k$, which is exactly what we need for gH to generate G/H.

b) $H \trianglelefteq G$ $[G:H] = m$. Prove $g^m \in H$ $\forall g \in G$.

Proof: Since $H \trianglelefteq G$, G/H is a group, and it has m elements, since H has index m in G. Let $g \in G$, so that $gH \in G/H$.

Then $|gH|$ is a divisor of m, by Langranges theorem. Thus, $(gH)^m = H$, since H is the identity element of G/H. But $(gH)^m = g^m H$; using our coset multiplication. Thus $g^m H = H$. Since $g^m = g^m \cdot e$ and $e \in H$, this means $g^m \in H$, as desired.

5. This problem deals with finite abelian groups.

   (a) (5 points) One element of order 12 in $\mathbb{Z}_4 \times \mathbb{Z}_{15} \times \mathbb{Z}_{18}$ is $(1, 5, 6)$. Find another one (no justification necessary).

   $(3, 10, 12)$

   (b) (5 points) Find a noncyclic subgroup of order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_{15} \times \mathbb{Z}_{18}$ (no justification necessary).

   $\{(0,0,0),\ (2,0,0),\ (0,0,9),\ (2,0,9)\}$

   (c) (5 points) Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_{15} \times \mathbb{Z}_{18}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ isomorphic? Why or why not?

   By Fundamental thm: $\cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \qquad \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_2 \times \mathbb{Z}_5$

   (pulled apart relatively prime parts).

   same lists, so yes isomorphic

   (d) (5 points) How many elements of $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_3$ have finite order? Why?

   6. Each coordinate must have finite order: 1 elements in $\mathbb{Z}$

   2 in $\mathbb{Z}_2$

   3 in $\mathbb{Z}_3$

   $1 \cdot 2 \cdot 3 = \boxed{6}$