MATH 113 MIDTERM
Thursday, October 11, 2012

This exam has 5+1 problems on 9 pages, including this cover sheet. The only thing you may have out during the exam is one or more writing utensils. You have 80 minutes to complete the exam.

The very last page is a bonus problem which is due by 12 noon at my office on Friday.

DIRECTIONS

- Be sure to carefully read the directions for each problem.
- All work must be done on this exam. If you need more space for any problem, feel free to continue your work on the back of the page or on the blank page at the end of the test. Draw an arrow or write a note indicating this, so I know where to look for the rest of your work.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- Good luck; do the best you can!

Problem	Max	Score
1	20	
2	30	
3	30	
4	10	
5	10	
B	10	
Total	100 (110)	

1. The parts of this problem are not related to each other. Your justifications should be very brief, and you don't need to use complete sentences.

   (a) (5 points) Suppose $G = \mathbb{Z}_8 \times \mathbb{Z}_6$ and $H = \langle (2,3) \rangle$. What is the order of the group G/H?

   $|G| = 48 \qquad |H| = 4$

   $|G/H| = 48/4 = \boxed{12}$

   (b) (5 points) What is the order of the element $(1,3,5,7,8)(3,4,5)(2,6)$ in S_9?

   $= (1,3,4,7,8)(5)(2,6)$

   order $= \mathrm{lcm}(2,5) = 10$

   (c) (5 points) Which finite abelian groups of order 18 have at least one element of order 9?

   $18 = 2 \cdot 3^2$

   2 groups $\boxed{\mathbb{Z}_2 \times \mathbb{Z}_9}$

   ~~$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$~~ No order 9 elt.

   (d) (5 points) If $\phi : G \to H$ is a group homomorphism, and $|G| = 6$ and $|H| = 2$, what are the possible sizes of $\ker \phi$?

   $G/\ker \cong \mathrm{im}$. im has size 1 or 2,

   so $6/|\ker| = 1$ or 2, $\boxed{|\ker| = 6 \text{ or } 3}$

2. (5 points each) For each of the following, the answer is worth 2 points, and the justification is worth 3 points. Circle the correct answer, and give a very brief justification of your answer (quote appropriate theorems, show relevant calculations, give a counterexample, etc.).

(a) The set of all 2×2 matrices with real entries forms a group under multiplication.

TRUE *FALSE*

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ not invertible.

(b) The group $\mathbb{Z}_8$ has a subgroup which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

TRUE *FALSE*

All subgroups of a cyclic gp are cyclic.

(c) If H is a subgroup of S_4, then H is also a subgroup of A_4.

TRUE *FALSE*

Counter example: $S_4 = H$.

or $H = \langle (1,2) \rangle$.

(d) If G is a group of order 35, then every proper subgroup of G is abelian.

TRUE *FALSE*

By Lagrange, proper subs can only have order 1, 5, or 7, so subs are trivial or $\cong \mathbb{Z}_5$ or $\mathbb{Z}_7$, since 5, 7 are prime.

(e) The groups $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ are isomorphic.

TRUE *FALSE*

Use FTFGAG:

$$\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24} \cong \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_8$$

$$\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40} \cong \mathbb{Z}_4 \times \ldots$$

No, since no $\mathbb{Z}_4$ factor above.

(f) If H is a normal subgroup of a group G and G/H is abelian, then G is abelian.

TRUE *FALSE*

$G = D_4$ not abelian

$H = \langle r \rangle = \{\text{rotations}\}$.

$G/H \cong \mathbb{Z}_2$. abelian

3. (5 points each) For each of the items listed below, give a SPECIFIC example with the stated property. Do not simply say why an example exists. All of these are possible.

(a) An infinite group G and a proper nontrivial normal subgroup H of G.

$G = \mathbb{Z}$

$H = 2\mathbb{Z}$

normal since G abelian.

(b) An element of $\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_6$ which has order 10 and does not have a 0 in any component.

$\boxed{(2, 1, 3)}$

order 2 5 2 , lcm = 10

(c) A proper nontrivial subgroup of $GL(3, \mathbb{R})$.

- diagonal matrices in $GL(3, \mathbb{R})$
- upper Δ " "
- lower Δ.

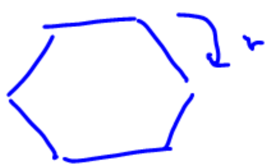
(d) An odd permutation with no fixed points in S_7. Write your answer in disjoint cycle notation or 2-line notation.

$$(1,2)(3,4,5,6,7)$$

odd + even = odd

(e) A subgroup of D_6 which is isomorphic to $\mathbb{Z}_3$.

$\langle r^2 \rangle$, where r denotes rotation by $60°$



(f) A nontrivial homomorphism from $\mathbb{Z}_3$ to $\mathbb{Z}_{24}$.

$\varphi: \mathbb{Z}_3 \longrightarrow \mathbb{Z}_{24}$

$0 \longmapsto 0$

$1 \longmapsto 8$

$2 \longmapsto 16$

i.e.

$\varphi(k) = 3k.$

4. (10 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) State and prove Lagrange's Theorem.

   (b) State Lagrange's Theorem. Use it to prove that every group of prime order p is cyclic.

See book

Pf: Suppose $|G| = p$, p prime. Then by Lagrange, any subgroup of G must have order 1 or p.

Let x be any non identity element of G. Then $\langle x \rangle$ has at least 2 elements, x and the identity e, so $|\langle x \rangle| = p$, and $\langle x \rangle$ is thus all of G, so G is cyclic.

5. (10 points) Prove **ONE** of the following. If you try both, clearly indicate which one you want to be graded.

   (a) Suppose that G is a group and $a, b \in G$. Prove that if $|ab| = k$, then $|ba| = k$.

   (b) Let G be an abelian group. Prove that the subset of all elements of order 3, together with the identity, is a subgroup of G.

a) Since $|ab| = k$, $(ab)^k = e$ and k is the smallest such element.

I.e. $\underbrace{(ab) \cdots (ab)}_{k \text{ copies}} = e$. Left mult by b, right mult by b^{-1}:

Then $b\underbrace{(ab) \cdots (ab)}_{k \text{ copies}} b^{-1} = bb^{-1}$, i.e.

$\underbrace{(ba) \cdots (ba)}_{k \text{ copies}} = e$, using associativity and mult. inverses.

Thus $|ba| \leq |ab| = k$. By a similar argument, $|ab| \leq |ba|$, so $|ba| = k$.

b) Note we can write our subset as $H = \{x \in G : x^3 = e\}$.

Clearly $e \in H$. Now, suppose $a, b \in H$. We WTS $ab^{-1} \in H$ to use the subgroup criterion.

Now $a^3 = e$ and $b^3 = e$, so we also have $e = (b^{-1})^3$ (by multiplying by b^{-1} 3 times on both sides.

Since G is abelian,

$$(ab^{-1})^3 = a^3 (b^{-1})^3 = e \cdot e = e,$$

So $ab^{-1} \in H$, and $H \leq G$.

6. This bonus question may be turned in at the end of the exam or submitted under my office door by 12 noon on Friday. You may use Fraleigh, your previous 113 homework, and your 113 notes if you like. You may **NOT** discuss this problem with any person, consult any other book or any website, or use any other resource you might imagine.

   Your goal is to find a subgroup H of S_{12} which meets the following two criteria:

   (a) $H \cong D_6 \times D_3$

   (b) No number in $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ is fixed by every permutation of H. (In other words, for each k, there exists a permutation in H for which k is not a fixed point.)

   An answer which meets the first criterion only will be worth 5 points. An answer which meets both criteria will be worth 10 points. Your justification will almost certainly use a picture.

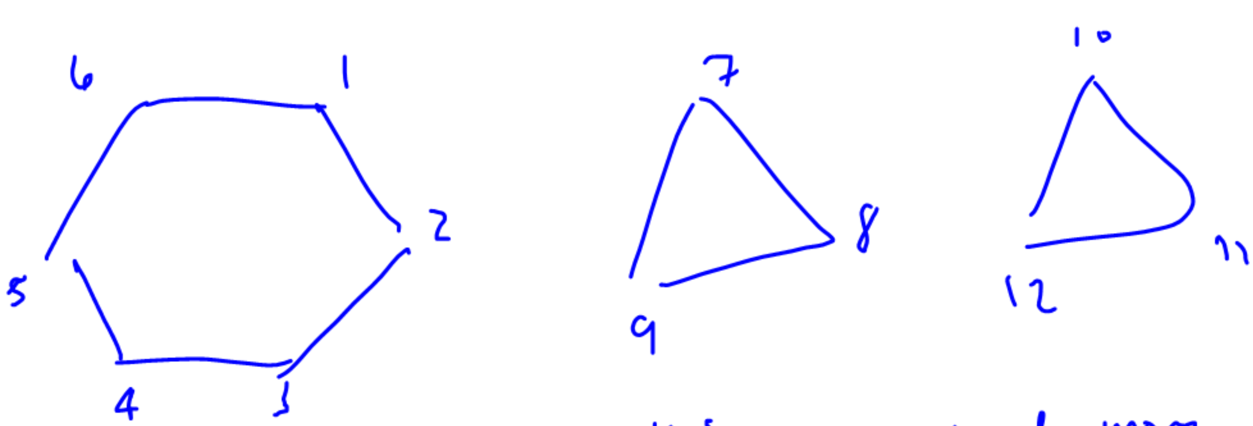
   You may use the following facts to help justify your answer:

   (i) Each dihedral group D_n is generated by a set of two elements: a rotation by $360/n$ degrees and any of the reflections.

   (ii) If a group G_1 is generated by a subset A of its elements and a group G_2 is generated by a subset B of its elements, then $A \times B = \{(a, b) : a \in A, b \in B\}$ is a generating set for $G_1 \times G_2$.

   This means it's sufficient to find a set of generators for $D_6 \times D_3$ and say which permutations those generators correspond to (instead of explicitly listing the isomorphism element by element for all 72 elements). If this page is not enough space (including the back), you are writing too much.

$$H = \langle (1,2,3,4,5,6),\ (1,2)(3,6)(4,5),\ (7,8,9)(10,11,12),\ (8,9)(11,12) \rangle$$



(real thing required more justification.)