MATH 104 MIDTERM
Thursday, March 17, 2016

This exam has 5 problems on 10 pages, including this cover sheet and extra blank space at the end. The only thing you may have out during the exam is the exam itself, your half sheet of notes, and one or more writing utensils. You have 80 minutes to complete the exam.

DIRECTIONS

- Be sure to carefully read the directions for each problem.
- All work to be graded must be done on this exam. If you want, you can tear off the last page, which is blank on both sides. If you do, put your name on it and turn it in with your exam. If you need extra space to finish a problem, make a note and finish it on the last page. This should really only happen if you mess up and want to start over without erasing.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- There is an extra challenge problem for strong students. If you finish early, you can get it from me when you turn in your exam. The challenge problem will only be graded if you already have an A on the exam without it. The problem involves proving a result we have not discussed together (and which is not in the textbook) and will only be used to help distinguish A grades from A+ grades.
- Good luck; do the best you can!

Problem	Max	Score
1	20	
2	25	
3	15	
4	20	
5	20	
Total	100	

Problem 0: Take a deep breath – you've got this!

1. (10 points each) Determine whether each of the following series converges or diverges. Your proofs may involve any results we have covered, including the series tests, but you must explicitly check that you meet the conditions to use a given theorem. Clearly mark where your scratch work ends and your proof begins.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n}{n!}$

Determine whether each of the following series converges or diverges. Your proofs may involve any results we have covered, including the series tests, but you must explicitly check that you meet the conditions to use a given theorem. Clearly mark where your scratch work ends and your proof begins.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(50 - \frac{2}{n}\right)$

2. (a) (2 points) Find the limit below. Show any computations you do, but a proof is not required until later in this problem.

$$\lim_{x\to 7^-} \frac{x+1}{x-7} =$$

(b) (4 points) Carefully state in terms of ϵ, δ, M, and/or N what it means for this limit to be what you claim it is. (Use these variables in the standard way we have done in class.)

On the following page, you will give a proof that your limit is correct, directly using the ϵ, δ, N, and or M version you stated above (no shortcut theorems). There is space for scratch work here.

(c) (14 points) Prove that (fill in your answer from the previous page):

$$\lim_{x \to 7^-} \frac{x+1}{x-7} =$$

Give your proof directly using the ϵ, δ, M, and/or N version of limits that you stated on the previous page (no limit shortcut theorems).

(d) (5 points) Is $f(x) = \frac{x+1}{x-7}$ uniformly continuous on the open interval $(8, 9)$? Briefly justify your answer.

3. (15 points) Suppose that (a_n) and (b_n) are convergent sequences of real numbers with $\lim a_n = A$ and $\lim b_n = B$. Prove that $(a_n + b_n)$ is a convergent sequence with $\lim a_n + b_n = A + B$. No limit shortcut theorems should be used in your proof. Clearly mark where your scratch work ends and your proof begins.

4. (2 points each) This page consists of TRUE/FALSE questions. No justification is required, though you may use any blank space for scratch work. (You will get 0 points for wrong or blank answers and 2 points for correct answers.) If you change your mind, make sure it is clear which is your final choice.

(a) The set of integers satisfies all the order axioms but not all of the field axioms.
TRUE **FALSE**

(b) If A and B are two nonempty subsets of $\mathbb{R}$, then $\inf(A \cup B) = \min\{\inf A, \inf B\}$.
TRUE **FALSE**

(c) If a and b are real numbers, then $|b| - |a| \leq |a - b|$.
TRUE **FALSE**

(d) If (a_n) is a sequence of real numbers such that $a_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then the sequence (a_n) converges.
TRUE **FALSE**

(e) If (a_n), (b_n), and (c_n) are sequences such that $a_n < b_n < c_n$ for all $n \in \mathbb{N}$ and the limits $\lim a_n$ and $\lim c_n$ both exist, then $\lim b_n$ also exists and $\lim a_n \leq \lim b_n \leq \lim c_n$.
TRUE **FALSE**

(f) The Bolzano-Weirstrass Theorem implies that the sequence (t_n) given by $t_n = n^{(-1)^n}$ has a convergent subsequence.
TRUE **FALSE**

(g) A monotone sequence can have more than one subsequential limit.
TRUE **FALSE**

(h) Every divergent sequence has at least two different subsequential limits.
TRUE **FALSE**

(i) The sequence $(\cos(\frac{n\pi}{6}))$ has precisely seven subsequential limits, all of which are real numbers.
TRUE **FALSE**

(j) The Root Test is inconclusive for the series $\sum(1 + \frac{1}{n})^n$.
TRUE **FALSE**

5. (2 points each) This page consists of more TRUE/FALSE questions. No justification is required, though you may use any blank space for scratch work. (You will get 0 points for wrong or blank answers and 2 points for correct answers.) If you change your mind, make sure it is clear which is your final choice.

(a) If $\sum t_n$ is a convergent series, then so is $\sum(-1)^n t_n$.
TRUE FALSE

(b) If f is a continuous function on the closed interval $[0, 10]$ with $f(0) = f(10)$, then there exists a pair of real numbers $a, b \in [0, 10]$ such that $|a-b| = \frac{5}{4}$ and $f(a) = f(b)$.
TRUE FALSE

(c) If f is a strictly increasing uniformly continuous function on an interval (a, b), then f is invertible and f^{-1} is uniformly continuous on its natural domain (i.e. the range of f).
TRUE FALSE

(d) The function $h(x) = \sqrt[3]{x}$ is uniformly continuous on $[4, \infty)$.
TRUE FALSE

(e) Let $p(x) = \frac{x+1}{x-1}$. Since $p(0) = -1$ and $p(2) = 3$, the Intermediate Value Theorem implies that p has a root in the interval $(0, 2)$.
TRUE FALSE

(f) If f is a function which is continuous on the open interval (a, b) and bounded on the same interval, then f is uniformly continuous on (a, b).
TRUE FALSE

(g) The function $x^2 \sin(\frac{1}{x})$ is continuous at $x = 0$.
TRUE FALSE

(h) We have $\lim x_n = L$, where $L \in \mathbb{R}$, if and only if for every $k \in \mathbb{N}$, there exists N such that $n > N$ implies $|x_n - L| < \frac{1}{k}$.
TRUE FALSE

(i) Suppose $f(x)$ is a uniformly continuous function on $\mathbb{R}$ and c is a constant in $\mathbb{R}$. Then $g(x) = f(x) + c$ and $h(x) = f(x + c)$ are both uniformly continuous on $\mathbb{R}$.
TRUE FALSE

(j) We have $\lim\limits_{x \to -\infty} f(x) = 17$ if and only if for every $\epsilon > 0$, there exists $N < 0$ such that $x < N$ implies $|f(x) - 17| < \epsilon$.
TRUE FALSE

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6. Challenge Problem: Prove the Strong Limit Comparison Test for series, stated below. (Note this can be used in more instances than the Math 1B version of the Limit Comparison Test, which is given in terms of the limit rather than limsup and liminf.)

   Suppose (a_n) and (b_n) are two sequences of positive real numbers. Show that if

$$0 < \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} < \infty,$$

   then

$$\sum_{k=1}^{\infty} a_k \text{ converges if and only if } \sum_{k=1}^{\infty} b_k \text{ converges.}$$

   Clearly distinguish between any scratch work and your actual proof.