MATH 104, HOMEWORK #12

Due Thursday, April 28

This is the last graded HW of the semester. Remember, consult the Homework Guidelines for general instructions. Results from class, our textbook, and graded homework are fair game to use unless otherwise specified. You may also use ungraded homework results from previous problem sets.

GRADED HOMEWORK:

1. Ross, Exercise 33.6. Then write an actual proof of Theorem 33.5 from p. 284. (You can use the book's outline, but fill in the details. You may also find that things are clearer if you reorganize the argument a bit.)

   Solution. Suppose f is integrable on $[a, b]$.

   For Exercise 33.5, we need to show that $M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S)$ for all subsets $S \subseteq [a, b]$.

   An elementary but tedious proof by cases goes as follows. Note that either $M(|f|, S) = M(f, S)$ or $M(|f|, S) = -m(f, S)$. Similarly, either $m(|f|, S) = m(f, S)$ or $m(|f|, S) = -M(f, S)$ or $m(|f|, S) = 0$. Checking each of the six pairings, we find that some pairings are incompatible, but for those which can simultaneously be true, our inequality holds.

   A slicker proof uses the book's hint. Since $f(x_0) \leq M(f, S)$ for all $x_0 \in S$ and $f(y_0) \geq m(f, S)$ for all $y_0 \in S$, we obtain the inequality $|f(x_0) - f(y_0)| \leq M(f, S) - m(f, S)$. Applying the Triangle Inequality (subtraction version), we then have

   $$|f(x_0)| - |f(y_0)| \leq |f(x_0) - f(y_0)| \leq M(f, S) - m(f, S)$$

   for all $x_0, y_0 \in S$. In other words, $M(f, S) - m(f, S)$ is an upper bound for the set $\{|f(x_0)| - |f(y_0)| : x_0, y_0 \in S\}$. I claim that actually $M(|f|, S) - m(|f|, S)$ is the least upper bound for this set. (It is very simple to verify that is is an upper bound, and you could give a quick contradiction proof that it is the least upper bound.) From this we immediately conclude $M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S)$.

   From here you should be able to quickly fill in details which will make the proof of Theorem 33.5 more readable.

   □

2. Ross, Exercise 33.7 and 33.8. (Both part of 33.8 should be quick corollaries once you have done the earlier bits.)

   Solution. Let f be a bounded function on $a, b]$, so there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

7(a) Show that $U(f^2,P) - L(f^2,P) \leq 2B[U(f,P) - L(f,P)]$ for all partitions P of $[a,b]$.

Let S be any subset of $[a,b]$ and let $x, y \in S$. We then have

$$\begin{aligned} f^2(x) - f^2(y) &= (f(x))^2 - (f(y))^2 \\ &= [f(x) + f(y)] \cdot [f(x) - f(y)] \\ &\leq 2B \cdot |f(x) - f(y)| \\ &\leq 2B \cdot [M(f,S) - m(f,S)]. \end{aligned}$$

Thus we see that $2B \cdot [M(f,S) - m(f,S)]$ is an upper bound for the set $\{f^2(x) - f^2(y) : x, y \in S\}$, and actually for the set $\{|f^2(x) - f^2(y)| : x, y \in S\}$. Using our observation from Problem 2, we notice that $M(f^2,S) - m(f^2,S)$ is the least upper bound of this set. Thus

$$M(f^2,S) - m(f^2,S) \leq 2B \cdot [M(f,S) - m(f,S)].$$

If $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a,b]$, note that the inequality above holds for all intervals $S = [t_{k-1}, t_k]$. Turning to the inequality we would like to prove, we now have

$$\begin{aligned} U(f^2,P) - L(f^2,P) &= \sum_{k=1}^{n} M(f^2,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) - \sum_{k=1}^{n} m(f^2,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} \left[M(f^2,[t_{k-1},t_k]) - m(f^2,[t_{k-1},t_k])\right] \cdot (t_k - t_{k-1}) \\ &\leq \sum_{k=1}^{n} 2B \cdot [M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])] \cdot (t_k - t_{k-1}) \\ &= 2B \sum_{k=1}^{n} [M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])] \cdot (t_k - t_{k-1}) \\ &= 2B[U(f,P) - L(f,P)] \end{aligned}$$

7(b) Show that if f is integrable on $[a,b]$, then f^2 is integrable on $[a,b]$.

Since f is integrable, it must be bounded. Choose $B > 0$ so that $|f(x)| \leq B$ for all $x \in [a,b]$. Let $\epsilon > 0$, which implies $\frac{\epsilon}{2B} < 0$ as well. Since f is integrable, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \frac{\epsilon}{2B}$. Then by part (a), $U(f^2,P) - L(f^2,P) \leq 2B \cdot [U(f,P) - L(f,P)] < 2B \cdot \frac{\epsilon}{2B} = \epsilon$, which implies that f^2 is integrable.

8(a) Suppose f and g are integrable on $[a,b]$. Show that fg is integrable on $[a,b]$.

Note that $4fg = (f+g)^2 - (f-g)^2$. We know that sums and constant multiples of integrable functions are integrable, so $f+g$ and $f-g$ are integrable. Then by the Exercise 33.7b above, $(f+g)^2$ and $(f-g)^2$ are integrable, and taking their difference, we see that $4fg$ is integrable. Since $fg = \frac{1}{4}4fg$, i.e. fg is a constant multiple of an integrable function, we see that fg is integrable on $[a,b]$.

8(b)Suppose f and g are integrable on $[a,b]$. Show that $\max(f,g)$ and $\min(f,g)$ are integrable on $[a,b]$.

We previously showed (in Section 17) that $\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ and $\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$. Since f and g are integrable, so are $f+g$ and $f-g$ (since sums and differences of integrable functions are integrable), and then so is $|f-g|$ (we just showed the absolute value of an integrable function is integrable). Finally, by the Section 17 formulas, $\max(f,g)$ and $\min(f,g)$ are both sums of integrable functions and therefore integrable on the same interval $[a,b]$.

$\square$

Notes. The lemma used in these two problems is nice enough that I probably should have mentioned it in class, but unless someone has a time machine, you are stuck with just these notes. I.e. $M(f,S) - m(f,S) = \sup\{f(x_1) - f(x_2) : x_1, x_2 \in S\} = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in S\}$. $\square$

3. Ross, Exercise 34.6. (Most likely you will need to use the Chain Rule. Be sure you check you meet the conditions to do so.) Let f be continuous on $\mathbb{R}$ and let

$$G(x) = \int_0^{\sin x} f(t)dt$$

for all $x \in \mathbb{R}$. Show G is differentiable on $\mathbb{R}$ and find $G'(x)$.

Solution. First, if we set $u(x) = \sin x$, then note that

$$G(x) = \int_{u(0)}^{u(x)} f(u)du.$$

We know that $u(x) = \sin x$ is differentiable on all of $\mathbb{R}$, and its derivative $u'(x) = \cos x$ is continuous on all of $\mathbb{R}$. Take I to be the open interval $I = (-2, 2)$. Notice $u(x) \in I$ for all I. Thus we meet the conditions to apply our Change of Variable Theorem. We conclude that the composition $(f \circ u)(x) = f(\sin x)$ is continuous on all of $\mathbb{R}$ and

$$G(x) = \int_{u(0)}^{u(x)} f(u)du = \int_0^x f(\sin x)\cos x dx.$$

Since $(f \circ u)' = \frac{d}{dx} f(\sin x) = f(\sin(x))\cos(x)$, we are now potentially in a good position to use the Fundamental Theorem. To verify we can do so, we just need to know that $f(\sin x)\cos x$

is integrable. This is indeed true – since f and $\sin x$ are continuous on all of $\mathbb{R}$, so is their composition. Then since $f(\sin x)$ and $\cos x$ are both continuous, so is their product. Thus $f(\sin x)\cos x$ is continuous on all of $\mathbb{R}$. Since continuous functions are integrable on all closed intervals contained in their domains, we see that $f(\sin x)\cos x$ is integrable on any interval $[a, b] \subseteq \mathbb{R}$.

Now, let's apply the Fundamental Theorem. Let $x_0 \in \mathbb{R}$, and take $[a, b]$ to be any closed interval such that $x_0 \in (a, b)$. Set

$$H(x) = \int_a^x f(\sin x)\cos x dx = \int_a^0 f(\sin x)\cos x dx + \int_0^x f(\sin x)\cos x dx = C + G(x)$$

for some constant C.

Then the Fundamental Theorem (Part II in our book) implies that $H(x)$ is continuous on all of $[a, b]$ and in particular at x_0. Furthermore, since $f(\sin x)\cos x$ is continuous at x_0, we have that H is differentiable at x_0 and $H'(x_0) = f(\sin x_0)\cos x_0$. Since this holds for every x_0 in $\mathbb{R}$, we see that H is continuous and differentiable on all of $\mathbb{R}$, and $H'(x) = f(\sin x)\cos x$.

Finally, since H and G differ by a constant, we see that G is also continuous and differentiable on all of $\mathbb{R}$ and $G'(x) = H'(x) = f(\sin x)\cos x$.

$\square$

Notes. I misspoke a bit in my parenthetical comment – you really wanted to verify you met the conditions to undo Chain Rule, i.e. Change of Variable/ u-Substitution. Notice our proof shows that it doesn't really matter which constant a we have as our lower limit in FTC II. $\square$

UNGRADED HOMEWORK:

Pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises.

Section	Exercises
33	1, 2*, 4, 9, 10, 13, 15ab
34	1, 2, 3, 5, 11, 12