MATH 104, HOMEWORK #11
Due Thursday, April 21

Remember, consult the Homework Guidelines for general instructions. Results from class, our textbook, and graded homework are fair game to use unless otherwise specified. You may also use ungraded homework results from previous problem sets.

GRADED HOMEWORK:

1. Ross, Exercise 29.13 and 29.14. Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$, and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x > 0$. Conclude that if f is differentiable on $\mathbb{R}$, $1 \leq f'(x) \leq 2$ for $x \in \mathbb{R}$, and $f(0) = 0$, then $x \leq f(x) \leq 2x$ for all $x > 0$.

   Solution. (13) By way of contradiction, assume that there exists some $b > 0$ with $f(b) > g(b)$. Define $h(x) = g(x) - f(x)$ for all $x \in \mathbb{R}$. Notice our assumption implies $h(b) < 0$, and we also have $h(0) = g(0) - f(0) = 0$. Since h is a difference of differentiable functions, it is also differentiable on $\mathbb{R}$. In particular, it is continuous on the closed interval $[0, b]$ and differentiable on the open interval $(0, b)$. Thus the Mean Value Theorem (for continuous functions) implies that there exists some number c in the interval $(0, b)$ such that

   $$h'(c) = \frac{h(b) - h(0)}{b - 0} = \frac{h(b)}{b} < 0.$$

   However, by our derivative shortcuts, $h'(x) = g'(x) - f'(x) \geq 0$ for all $x \in \mathbb{R}$, so we have arrived at a contradiction.

   (14) Let $d(x) = x$ and $g(x) = 2x$. Since $d'(x) = 1$ and $g'(x) = 2$ for all $x \in \mathbb{R}$ and $d(0) = f(0) = g(0)$, we may simply apply the result above for the pair d and f and the pair f and g, yielding $x \leq f(x) \leq 2x$. □

   Notes. This is about the level/length of an easy-medium proof on the final. □

2. Ross, Exercise 29.18. Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Select $s_0 \in \mathbb{R}$, and set $s_n = f(s_{n-1})$ for $n \geq 1$. Prove that (s_n) converges and f has a fixed point.

   Solution. (a) First, let's show that (s_n) converges to 0. Following the book's hint, we will first show that $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}|$ for $n \geq 1$. First of all, if there exists N such that $s_N - s_{N-1}$, i.e. $f(s_{N-1}) = s_{N-1}$, then $s_{N+1} = f(s_N) = f(s_{N-1}) = s_N$, and actually $s_m = s_N$ for all $m > N$ and $\lim s_n = s_N$, since our sequence is eventually constant. Otherwise $s_n - s_{n-1} \neq 0$ for all n. Then

   $$\frac{|s_{n+1} - s_n|}{|s_n - s_{n-1}|} = \frac{|f(s_n) - f(s_{n-1})|}{|s_n - s_{n-1}|}.$$

   Since $s_n \neq s_{n-1}$, one of (s_n, s_{n-1}) and (s_{n-1}, s_n) is a valid open interval I. Note f is differentiable on I and defined on the closure of I (i.e. $I \cup \{s_n\} \cup \{s_{n-1}\}$), so we may apply the Mean Value Theorem. Thus for any n, there exists x_n with

   $$f'(x_n) = \frac{s_{n+1} - s_n}{s_n - s_{n-1}}.$$

Then

$$\frac{|s_{n+1} - s_n|}{|s_n - s_{n-1}|} = |f'(x_n)| \leq a,$$

or $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}|$ for $n \geq 1$. Applying this recursively, we see that $|s_n - s_{n-1}| \leq a^{n-1}|s_1 - s_0|$. Using this fact along with the triangle inequality yields, for $m > n$,

$$\begin{aligned} |s_m - s_n| &\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n| \\ &\leq (a^{m-1} + a^{m-2} + \cdots + a^n)|s_1 - s_0| \\ &\leq \frac{a^n}{1-a}|s_1 - s_0| \end{aligned}$$

Finally we can show (s_n) is a Cauchy sequence. Let $\epsilon > 0$ and take $N \log_a[\frac{(1-a)\epsilon}{|s_1 - s_0|}]$. Let $n > N$. Since we have our base satisfying $0 < a < 1$, this implies $a^n < a^N = \frac{(1-a)\epsilon}{|s_1 - s_0|}$. Thus, for $m > n > N$, we have

$$|s_m - s_n| \leq \frac{a^n}{1-a}|s_1 - s_0| < \frac{\frac{(1-a)\epsilon}{|s_1 - s_0|}}{(1-a)}|s_1 - s_0| = \epsilon.$$

Thus (s_n) is a Cauchy sequence and therefore convergent.

(b) Let $L = \lim_{n\to\infty} s_n$, where $L \in \mathbb{R}$ by part (a). Consider the sequence $f(s_n)$, which is precisely the sequence (s_{n+1}). Thus $\lim f(s_n) = \lim s_{n+1} = \lim s_n = L$. Since f is continuous on $\mathbb{R}$, $\lim s_n = L$ implies $\lim f(s_n) = f(L)$. Thus we have $L = f(L)$, since limits are unique when they exist, and f has a fixed point. □

Notes. Part (b) is rather slick here, but the whole problem nicely ties together many topics we have discussed. □

3. Ross, Exercise 32.6. Let f be a bounded function on $[a, b]$. Suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums such that $\lim(U_n - L_n) = 0$. Show that f is integrable and $\int_a^b f = \lim U_n = \lim L_n$.

Solution. Notice that in our Darboux sums U_n and L_n, the function is the same throughout, so all that is changing is our partition of $[a, b]$. Let $U_n = U(f, P_n)$ and $L_n = L(f, Q_n)$ for all n, where P_n and Q_n are the appropriate partitions of $[a, b]$. To show f is integrable, we will show that for each $\epsilon > 0$ there exists a partition R for which $U(f, R) - L(f, R) < \epsilon$.

Let $\epsilon > 0$. Since $\lim(U_n - L_n) = 0$, there exists an integer N such that $n > N$ implies $U_n - L_n < \epsilon$. In particular this holds for $N + 1$. Take R to be the partition $P_{N+1} \cup Q_{N+1}$. Then $U(f, R) \leq U(f, P_{N+1}) = U_{N+1}$ since R refines P_{N+1}. Similarly, $L(f, R) \geq L(f, Q_{N+1}) = L_{N+1}$, or equivalently, $-L(f, R) \leq -L(f, Q_{N+1}) = -L_{N+1}$. Thus $U(f, R) - L(f, R) \leq U_{N+1} - L_{N+1} < \epsilon$, so f is integrable.

Rearranging the inequality $L_n \leq L(f) = \int_a^b f = U(f) \leq U_n$, we see that $0 \leq (\int_a^b f) - L_n \leq U_n - L_n$. Since $n > N$ implies $U_n - L_n < \epsilon$, we have $|(\int_a^b f) - L_n| < \epsilon$, which is exactly what we need to show that $\lim L_n = \int_a^b f$. The proof that $\lim U_n = \int_a^b f$ is similar. □

Notes. Note that for a given n, you cannot assume that U_n and L_n are using the same partition, which is why we had to take the union. Also note that we cannot separate $\lim(U_n - L_n)$ into $\lim U_n - \lim L_n$ unless we know these limits exist. $\square$

UNGRADED HOMEWORK:

Pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises.

Section	Exercises
29	1, 2, 3, 4, 7*, 8, 9, 10, 11, 12, 17
30	1, 2, 3
32	1, 2, 3, 4*, 5*, 8*