MATH 104, HOMEWORK #10

Due Thursday, April 14

Remember, consult the Homework Guidelines for general instructions. Results from class, our textbook, and graded homework are fair game to use unless otherwise specified. You may also use ungraded homework results from previous problem sets.

GRADED HOMEWORK:

1. Ross, Exercise 25.12. If $\sum_{k=1}^{\infty} g_k$ is a series of continuous functions on $[a, b]$ that converges uniformly to g on $[a, b]$, then

$$\int_a^b g(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx.$$

Solution. We need to show that the series of integrals on the right converges to the integral on the left. From a section 25 theorem, we know that we may interchange integrals and limits for a uniformly convergent series, so

$$\begin{aligned}\int_a^b g(x)dx &= \int_a^b \left(\sum_{k=1}^{\infty} g_k(x)\right) dx \\ &= \int_a^b \left(\lim_{n\to\infty} \sum_{k=1}^{n} g_k(x)\right) dx \\ &= \lim_{n\to\infty} \int_a^b \left(\sum_{k=1}^{n} g_k(x)\right) dx.\end{aligned}$$

For finite sums, we may interchange integrals and sums (by induction on the number of terms, using $\int(f+h) = \int f + \int h$ as the base case). Thus, we may continue rewriting:

$$\begin{aligned}\lim_{n\to\infty} \int_a^b \left(\sum_{k=1}^{n} g_k(x)\right) dx &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \int_a^b g_k(x)dx\right) \\ &= \sum_{k=1}^{\infty} \int_a^b g_k(x)dx\end{aligned}$$

□

Notes. Note someone asked on Piazza if the result $\int(f+h) = \int f + \int h$ was fair game, which I confirmed. Note an important fact here – we have potential issues with swapping integrals and infinite sums, and the uniform convergence is critical for ensuring this is sensible. (Just as we had potential issues with commutativity and associativity for infinite sums of numbers.) Theorems that we have for finite sums do not automatically pass through to infinite sums by taking limits. □

2. Ross, Exercise 28.4abc and 28.5c. Let $f(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$.

Solution. 28.4(a) First we will show f is differentiable at each $a \neq 0$. Using, our derivative shortcuts – all of which apply when $a \neq 0$ since x^2, $\sin x$, and $\frac{1}{x}$ are all differentiable then, with no zero denominators – we have for $x \neq 0$ (applying in order: product rule, then chain rule, then shortcut for power functions or quotient rule):

$$\begin{aligned}
f'(x) &= 2x\sin\left(\frac{1}{x}\right) + x^2 \cdot \frac{d}{dx}\sin\left(\frac{1}{x}\right) \\
&= 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\frac{1}{x} \\
&= 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right) \cdot \frac{d}{dx}x^{-1} \\
&= 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right) \cdot (-x^{-2}) \\
&= 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).
\end{aligned}$$

Thus $f'(a) = 2a\sin\left(\frac{1}{a}\right) - \cos\left(\frac{1}{a}\right)$ when $a \neq 0$.

(b) Using the definition, if it exists, we have

$$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{f(x)}{x}.$$

Since this limit only considers points *near* $x = 0$ (and not *at* $x = 0$), we have

$$f'(0) = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{x^2\sin(\frac{1}{x})}{x} = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right).$$

Since $\sin\left(\frac{1}{x}\right)$ is bounded, this limit is 0, so $f'(0) = 0$.

(c) For f' to be continuous at $x = 0$, we would need $\lim_{x\to 0} f'(x) = f'(0) = 0$. We will show this does not hold by exhibiting a sequence $(x_n) \to 0$ with $(f(x_n))$ not limiting to 0. Take $x_n = \frac{1}{n\pi}$. Then $f(x_n) = \frac{2}{n\pi}\sin(n\pi) - \cos(n\pi)$. The first term limits to 0, but the second oscillates between 1 and -1, so we see that $\lim f(x_n)$ does not exist, and thus f' is not continuous at $x = 0$.

28.5(c) The limit $\lim_{x\to 0}\frac{g(f(x)) - g(f(0))}{f(x) - f(0)}$ is meaningless because no matter what $\delta > 0$ we take, the interval $(-\delta, \delta)$ around $x = 0$ contains infinitely many points with $f(x) = 0$. (More specifically, we could take all $x = \frac{1}{n\pi}$ with n large enough.) Thus there is no interval around $x = 0$ on which the fraction is actually defined.

$\square$

Notes. Note that in the last step of part (b), you need to check that the second factor's absolute value is not going off to infinity. It is NOT sufficient to have one factor limiting to 0, as we have previously seen. $\square$

3. Ross, Exercise 28.8. Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.

Solution. (a) Here we need $\lim_{x\to 0} f(x) = f(0) = 0$. We will do a $\delta - \epsilon$ proof. Let $\epsilon > 0$ and choose $\delta = \sqrt{\epsilon}$. Let $|x - 0| < \delta$. If x is irrational, then $|f(x) - f(0)| = |0 - 0| = 0 < \epsilon$. If x is rational, then $|f(x) - f(0)| = |x^2 - 0| = x^2 < (\delta)^2 = (\sqrt{\epsilon})^2 = \epsilon$. Either way, we satisfy $|f(x) - f(0)| < \epsilon$, so our limit is 0, and thus f is continuous at $x = 0$.

(b) We will show that f is discontinuous everywhere else. First, suppose $x \neq 0$ is rational, so $f(x) = x^2 > 0$. The irrationals are dense in $\mathbb{R}$, so every open interval contains an irrational number. Let $\epsilon = \frac{x^2}{2}$ and $\delta > 0$. Let a be an irrational number in the interval $(-\delta, \delta)$. Then $|f(x) - f(a)| = |x^2 - 0| = x^2 > \frac{x^2}{2} = \epsilon$, so f is not continuous at our rational x.

Next, suppose x is irrational. The rationals are dense in $\mathbb{R}$, so every open interval contains a rational number. Let $\epsilon = x^2$ and $\delta > 0$. Let q be a rational number in the interval $(x-\delta, x+\delta)$ such that $f(q) = q^2 > x^2$. (If $x > 0$, we take q from the right half of the interval, and if $x < 0$, we take q from the left half od the interval.) Then $|f(x) - f(q)| = |0 - q^2| = q^2 > x^2 = \epsilon$ and thus f is not continuous at our irrational x.

(c) Here we need $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ to be a real number; we claim it is 0. Let $\epsilon > 0$ and take $\delta = \epsilon$. Suppose $|x - 0| < \delta = \epsilon$. Then for rational x, we have $|\frac{f(x)}{x} - 0| = |\frac{x^2}{x}| = |x| < \epsilon$. For irrational x, we have $|\frac{f(x)}{x} - 0| = |\frac{0}{x} - 0| = |0 - 0| < \epsilon$. Either way, $|\frac{f(x)}{x} - 0| < \epsilon$, so our limit is 0, as we claimed, and thus $f'(0) = 0$.

$\square$

Notes. For part (b), you could instead have built sequences with $\lim x_n = x$ but $\lim f(x_n) \neq f(x)$. Part (c) would likely be quite hard if you didn't figure out what number $f'(0)$ is. $\square$

* Extra Credit (3 points): Ross, Exercise 28.15.

UNGRADED HOMEWORK:

Pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises.

Section	Exercises
25	2, 4*, 5*, 7, 8, 9*, 10,
26	2, 3, 4, 5, 6, 7
28	2, 3, 9, 11*, 14*