MATH 104, HOMEWORK #7

Due Thursday, March 10

Remember, consult the Homework Guidelines for general instructions. Results from class, our textbook, and graded homework are fair game to use unless otherwise specified. You may also use ungraded homework results from previous problem sets.

GRADED HOMEWORK:

1. (a) Ross, Exercise 18.10.

   (b) For which k with $0 < k < 2$ must there exist $x, y \in [0, 2]$ so that $|y - x| = k$ and $f(x) = f(y)$? Justify your answer.

Solution. (a) Suppose $f(x)$ is continuous on the closed interval $[0, 2]$ and $f(0) = f(2)$. Note that $f(x + 1)$ is defined on $[-1, 1]$ (as it is simply a translation of $f(x)$), and it is continuous on that interval. Define $g(x) = f(x + 1) - f(x)$, and note that g is defined on the closed interval $[0, 1]$, and since it is the difference of continuous functions defined on that domain, it is continuous as well. Observe that $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1) = f(0) - f(1)$, so $g(0) = -g(1)$. We now consider two cases. If $g(0) = 0$, then $f(1) = f(0)$, and we have found two inputs $x = 0$ and $y = 1$ satisfying $|x - y| = 1$ and $f(x) = f(y)$. Otherwise, one of $g(0)$ and $g(1)$ is positive and the other is negative. Applying the Intermediate Value Theorem, we see there must exist a value $a \in (0, 1)$ with $g(a) = 0$. In this case, we then have $0 = g(a) = f(a+1) - f(a)$, so $f(a) = f(a + 1)$. Thus we can take $x = a$ and $y = a + 1$ (note both are in the interval $[0, 2]$) satisfying $|x - y| = 1$ and $f(x) = f(y)$.

(b) Claim: if $k > 1$, no such points are guaranteed. For a counterexample, take any continuous function with $f(0) = f(1) = f(2)$ which is strictly negative on $(0, 1)$ and strictly positive on $(1, 2)$. Then any x, y with $f(x) = f(y) \neq 0$ will lie entirely in $(0, 1)$ or entirely in $(1, 2)$.

It seems like the result might hold for any $k < 1$, but it is rather tough to prove and definitely not even obviously true. A reasonable partial answer might go as follows: We can at least get an infinite set of small k values as follows. It is easy to see that the proof in part (a) can be generalized – if $f(x_1) = f(y_1)$ and $|x_1 - y_1| = d$, the same technique can be used to show that there exist points x_2 and y_2 with $f(x_2) = f(y_2)$ and $|x_2 - y_2| = \frac{d}{2}$. Repeating as necessary, we can show that the result holds for any $k = \frac{1}{2^n}$.

You can show that more rational numbers $k < 1$ work as follows. Suppose $k = \frac{2}{n}$ for some integer $n \geq 1$. Take $g(x) = f(x + k) - f(x)$, which will be defined on the interval $[0, 2 - k]$. Then consider $g(0) = f(k) - f(0)$, $g(k) = f(2k) - f(k)$, $g(2k) = f(3k) - f(2k)$, etc. up to $g(2 - k) = f(2) - f(2 - k)$. Note that these sum to $f(2) - f(0) = 0$. Thus, you can convince yourself that you must have $g = 0$ at one of these points, in which case you immediately have a pair that works, or you have an adjacent pair with $g > 0$ for one and $g < 0$ for the other, in which case you can apply the Intermediate Value Theorem as before to guarantee a spot for which $g = 0$. $\square$

Notes. Make sure you get part (a) on this one, as well as how to generalize it to $k = \frac{1}{2^n}$. The rest is mostly an exploration problem to help you practice examining your proofs to see if they generalize, possibly with a few modifications. I'm not sure if we have the right tools to give a complete proof yet, but if you discovered one, excellent!

Update: Some very clever students were able to show that actually these are all the k-values that work, so it looks like the fractions $k = \frac{2}{n}$ with $n \in \mathbb{Z}^+$ are all we have. (This is not quite what I expected to happen!) □

2. (a) Complete Ross, Exercise 19.4.
   (b) What are the pros and cons of the three proofs that $f(x) = \frac{1}{x^2}$ is not uniformly continuous? (The three proofs being the one you just wrote, plus Example 3 and Example 6 in Section 19.)

Solution. (a) We will prove 19.4 by contradiction. Assume f is uniformly continuous on a bounded set S, but f is not bounded on S. Since f is not bounded on S, given any $M > 0$, there exists some $x \in S$ such that $|f(x)| > M$ (otherwise $\pm M$ would be bounds for f). With this in mind, construct a sequence of elements in S: for each $n \in N$, choose some $x_n \in S$ such that $|f(x_n)| > M$. Since each term of the sequence (x_n) is in S, our sequence is bounded, and thus by Bolzano-Weirstrass, it has a convergent subsequence (x_{n_k}). As convergent subsequences are always Cauchy, we see that (x_{n_k}) is a Cauchy sequence. Since uniformly continuous functions map Cauchy sequences to Cauchy sequences, we see that $(f(x_{n_k}))$ must be a Cauchy sequence of real numbers and must therefore converge to some real number, i.e. $\lim_{k\to\infty} f(x_{n_k}) = L \in \mathbb{R}$. However, this is incompatible with the fact that $|f(x_{n_k})| > n_k$ for all $k \in \mathbb{N}$, which implies $\lim_{k\to\infty} |f(x_{n_k})| = \infty$.

(b) Since $\frac{1}{x^2}$ is unbounded on the interval $(0, 1)$, we see that it cannot possibly be uniformly continuous, by the contrapositive of the result in (a).

□

Notes. We have essentially done this proof several times now with slightly different assumptions in the last few sections; you should have a good sense of how it goes. Only parts (a) and (b) from the textbook were graded. The pros/cons portion was to get you thinking about efficiency, since we have a good handful of ways to deal with uniform continuity, and you want to have a good sense for which method to pick when it's up to you on an exam. You should have noted that the Example 3 proof was very technical but relied only on definitions (and experience with $\delta - \epsilon$ proofs), while the proofs in Example 6 and part (b) above were very quick once we had a couple easily stated results. (When you are lucky, the more machinery you have at your disposal, the shorter your proofs can get.) □

3. Complete Ross, Exercise 19.1. (Only a few of these will be carefully graded, but you won't know which ahead of time. Make sure each part is clearly labeled.) You can use any theorems through Section 19.

Solution. (a) This IS uniformly continuous on $[0, 2\pi]$. This function is made of sums, differences, products, and compositions of functions which are continuous on all of $\mathbb{R}$ (and thus on $[0, 2\pi]$), so it is continuous on $[0, 2\pi]$. The theorem stating that continuous functions on closed intervals are uniformly continuous completes the proof.

(c) This function IS uniformly continuous on $(0, 1)$, as it can be extended to $\tilde{f}(x) = x^3$ on $[0, 1]$, which is uniformly continuous since it is continuous on a closed interval.

(d) This function is NOT uniformly continuous on $\mathbb{R}$. We will prove this one from the (negation of) the definition. Pick $\epsilon = 1$. We will show that for all $\delta > 0$, there exist $x_1 = x + \frac{\delta}{3}$ and $x_2 = x$ in $\mathbb{R}$ with $|x_1 - x_2| = |\frac{\delta}{3}| < \delta$ but $|x_1^3 - x_2^3| = |(x + \frac{\delta}{3})^3 - x^3| \geq \epsilon = 1$. Let $\delta > 0$. Expand $|(x + \frac{\delta}{3})^3 - x^3|$ as

$$\left| x^3 + 3\frac{\delta}{3}x^2 + 3(\frac{\delta}{3})^2 x + (\frac{\delta}{3})^3 - x^3 \right| = \left| \delta x^2 + \frac{\delta^2}{3}x + \frac{\delta^3}{27} \right|.$$

Since we have a parabola opening upward (most importantly, it is an unbounded function), there is definitely a large enough choice of x for which

$$\left| \delta x^2 + \frac{\delta^2}{3}x + \frac{\delta^3}{27} \right| = \delta x^2 + \frac{\delta^2}{3}x + \frac{\delta^3}{27} \geq 1.$$

(g) This function IS uniformly continuous on $(0, 1]$. He we take an extension of $f(x) = x^2 \sin(\frac{1}{x})$ on $(0, 1]$ to a continuous function $\tilde{f}(x)$ on $[0, 1]$ by setting

$$\tilde{f}(x) = \begin{cases} x^2 \sin(\frac{1}{x}), \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases}$$

It is clear that $\tilde{f}$ is continuous at points other than $x = 0$, since it is made of compositions, products, and quotients of appropriate continuous functions. It remains to check that $\tilde{f}$ is continuous at $x = 0$. Let $\epsilon > 0$, and choose $\delta = \sqrt{\epsilon}$. Then $|x| = |x - 0| < \delta$ implies that $|x|^2 = |x^2| < \delta^2$. Notice that $|\tilde{f}(x) - 0| \leq |x^2|$ for all $x \in [0, 1]$ – it is obvious for $x = 0$ and otherwise we have $|\tilde{f}(x) - 0| = |x^2 \sin(\frac{1}{x})| \leq |x^2|$, since $|\sin(\frac{1}{x})| \leq 1$. Using transitivity to string everything together, we have $|\tilde{f}(x) - f(0)| \leq |x^2| < \delta^2 = \epsilon$, which implies that $\tilde{f}$ is continuous at $x = 0$ and thus on all of $[0, 1]$.

Thus, we have found an extension of our function which is continuous on a closed interval containing $(0, 1]$, so f is uniformly continuous.

□

Notes. Parts (a), (c), and (g) were graded. The part (d) solution is just there to give you an example of a reasonable proof that something is not uniformly continuous.

In part (g), note that you could also have seen that f was continuous by expressing f as the product of x and the function in Example 7 of Section 19, but the proof that the Example 7 function is continuous at 0 needs a proof, which is Exercise 17.9. Normally only ungraded

HW from previous sets is considered fair game without proof (as stated in the directions at the top), but since I gave you the Section 17 ungraded HW last week, I would accept it on this problem. □

UNGRADED HOMEWORK:

Pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises. Note I already listed the first two sections of ungraded HW on Piazza last week; they are here again in case you are using your homework sheets to keep track of them.

Section	Exercises
15	1, 2, 3, 4, 5, 6, 7
17	1, 2, 3, 5, 8, 9, 10, 11
18	1, 2, 3, 4, 5, 6, 7, 8, 11,
19	2, 3, 5, 6, 8, 9