MATH 104, HOMEWORK #6
Due Thursday, February 25

Remember, consult the Homework Guidelines for general instructions. Results from class, our textbook, and graded homework are fair game to use unless otherwise specified. You may also use ungraded homework results from previous problem sets.

GRADED HOMEWORK:

1. Ross, Exercise 12.12.

Solution. We will first show that $\limsup \sigma_n \leq \limsup s_n$, using the book's hint, i.e. to first show that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + \cdots + s_N) + \sup\{s_n : n > N\}.$$

We start with the LHS:

$$\begin{aligned}
\sup\{\sigma_n : n > M\} &= \sup\left\{\frac{1}{n}(s_1 + \cdots + s_n) : n > M\right\} \\
&= \sup\left\{\frac{1}{n}(s_1 + \cdots + s_N) + \frac{1}{n}(s_{N+1} + \cdots + s_n) : n > M\right\} \\
&\leq \sup\left\{\frac{1}{n}(s_1 + \cdots + s_N) : n > M\right\} + \sup\left\{\frac{1}{n}(s_{N+1} + \cdots + s_n) : n > M\right\},
\end{aligned}$$

with the last bit using the fact that

$$\sup\{x_n + y_n : n > N\} \leq \sup\{x_n : n > N\} + \sup\{y_n : n > N\}.$$

(This is covered in the ungraded HW, Exercise 12.4; I'll leave the proof to you.) Continuing, we have

$$\begin{aligned}
\sup\{\sigma_n : n > M\} &\leq \sup\left\{\frac{1}{n}(s_1 + \cdots + s_N) : n > M\right\} + \sup\left\{\frac{1}{n}(s_{N+1} + \cdots + s_n) : n > M\right\} \\
&\leq \sup\left\{\frac{1}{M}(s_1 + \cdots + s_N)\right\} + \sup\left\{\frac{1}{n}(s_{N+1} + \cdots + s_n) : n > M\right\}, \\
&\quad \text{since } n > M \text{ implies } \frac{1}{n} < \frac{1}{M}, \\
&= \frac{1}{M}(s_1 + \cdots + s_N) + \frac{n-N}{n}\sup\{s_n : n > N\} \\
&< \frac{1}{M}(s_1 + \cdots + s_N) + \sup\{s_n : n > N\}, \text{ since } 0 < \frac{n-N}{n} < 1.
\end{aligned}$$

Note that we needed the s_n to be nonnegative to get the second line in the block above.

So we have $\sup\{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + \cdots + s_N) + \sup\{s_n : n > N\}$ for any $M > N$. Keeping N fixed and taking the limit as M goes to infinity, we obtain

$$\begin{aligned}
\limsup \sigma_n &\leq \lim_{M\to\infty} \frac{1}{M} \cdot \lim_{M\to\infty}(s_1 + \cdots + s_N) + \lim_{M\to\infty} \sup\{s_n : n > N\} \\
&= 0 \cdot \lim_{M\to\infty}(s_1 + \cdots + s_N) + \sup\{s_n : n > N\},
\end{aligned}$$

with the last term due to the fact that $\sup\{s_n : n > N\}$ does not rely on M as all (so it can be treated as a constant with respect to M). Finally, we take the limit as N goes to ∞, obtaining

$$\lim_{N\to\infty} \limsup \sigma_n \leq \lim_{N\to\infty} \sup\{s_n : n > N\},$$

which simplifies to

$$\limsup \sigma_n \leq \limsup s_n.$$

WE STILL NEED TO SHOW THE LIMINF PART.

The middle inequality, i.e. $\liminf \sigma_n \leq \limsup \sigma_n$, holds for any sequence. Thus, by transitivity, we have

$$\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$$

□

Notes. This one was a bit brutal. In particular, though I could get the book's hint to work for lim sup, I kept running into snags trying to get the symmetric proof for lim inf. There are a few ways to get the lim inf half, and in the end I decided the one I wrote up was probably simplest. Be extremely careful – in discussions with multiple students, we found a number of wrong proofs on the internet. □

2. Ross, Exercise 14.6.

Solution. (a) We want to show that if $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges, using the Cauchy Criterion. That is, using version 3, we need to show that for each $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|\sum_{k=m}^{n} a_k b_k| < \epsilon$.

Let $\epsilon > 0$, and suppose $B > 0$ is a real number with $|b_n| < B$ (such a number exists since (b_n) is bounded). Since $\sum |a_n|$ converges, it satisfies the Cauchy Criterion and thus there exists a number N so that for all $m, n > N$ we have

$$\sum_{k=m}^{n} |a_k| = \left|\sum_{k=m}^{n} |a_k|\right| < \frac{\epsilon}{B},$$

since $\frac{\epsilon}{B} > 0$. Assume $m, n > N$. Now, applying the triangle inequality ($n - m$ times) and the above information, we have

$$\begin{aligned}
\left|\sum_{k=m}^{n} a_k b_k\right| &\leq \sum_{k=m}^{n} |a_k b_k| \\
&= \sum_{k=m}^{n} |a_k| \cdot |b_k| \\
&\leq \sum_{k=m}^{n} |a_k| \cdot B \\
&= B \sum_{k=m}^{n} |a_k| \\
&< B \cdot \frac{\epsilon}{B} \\
&= \epsilon,
\end{aligned}$$

which shows that $\sum a_n b_n$ satisfies version 3 of the Cauchy Criterion and therefore converges.

(b) Next we conclude that absolutely convergent sequences converge, as a special case of part (a). Suppose (a_n) is an absolutely convergent sequence, so that $\sum |a_n|$ converges. Let (b_n) be the constant sequence with $b_n = 1$ for all n; it is clearly bounded above and below by 1. By part (a), $\sum a_n b_n = \sum a_n$ converges. $\square$

Notes. This is a pretty reasonable difficulty for an exam problem. You have to know definitions and basics like the triangle inequality, but the proof itself is not too crazy. $\square$

3. Using the Cauchy criterion directly (and none of the later tests for convergent series), show that one of the following series converges and one diverges. You may use whichever of the three versions is most convenient

$$\sum \frac{1}{n^2} \qquad \text{and} \qquad \sum \left(\sqrt{n+1} - \sqrt{n}\right)$$

Solution. First we will show $\sum \frac{1}{n^2}$ converges. Using version 3 of the Cauchy Criterion, we need to show that for all $\epsilon > 0$, there exists a number N so that $m, n > N$ implies $\left| \sum_{k=m}^{k} \frac{1}{k^2} \right| < \epsilon$.

Let $\epsilon > 0$, and take $N = \frac{2}{\epsilon}$, i.e. $\epsilon = \frac{2}{N}$. Note that for any integer k, we have $\frac{1}{k^2} < \frac{1}{k(k+1)}$ (larger denominator), and also $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ Then

$$\begin{aligned}
\left| \sum_{k=m}^{k} \frac{1}{k^2} \right| &= \sum_{k=m}^{k} \frac{1}{k^2} \\
&< \sum_{k=m}^{n} \frac{1}{k(k+1)} \\
&= \sum_{k=m}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \frac{1}{m} - \frac{1}{n+1} \\
&= \left| \frac{1}{m} - \frac{1}{n+1} \right| \\
&\le \left| \frac{1}{m} \right| + \left| -\frac{1}{n+1} \right| \\
&= \left| \frac{1}{m} \right| + \left| \frac{1}{n+1} \right| \\
&\le \frac{1}{N} + \frac{1}{N} = \frac{2}{N} = \epsilon
\end{aligned}$$

Next, we'll show that $\sum \left(\sqrt{n+1} - \sqrt{n}\right)$ diverges by showing it violates the Cauchy Criterion. In particular, we will show that if we take $\epsilon = 1$, then for every positive number N, there is a choice of $m, n > N$ with $\left| \sum_{k=m}^{n} \left(\sqrt{k+1} - \sqrt{k}\right) \right| \ge 1 = \epsilon$.

Set $\epsilon = 1$ and let N be any positive number. Let x and y be different positive integers so that $x^2 > y^2 > N$ (such integers clearly exist since there are infinitely many perfect squares). If we set $n + 1 = x^2$ and $m = y^2$, then

$$\begin{aligned}\left|\sum_{k=m}^{n}\left(\sqrt{k+1}-\sqrt{k}\right)\right| &= \sum_{k=m}^{n}\left(\sqrt{k+1}-\sqrt{k}\right)\\ &= \sqrt{n+1}-\sqrt{m}\\ &= \sqrt{x^2}-\sqrt{y^2}\\ &= x-y \geq \epsilon = 1,\end{aligned}$$

with $x - y \geq 1$ since they are distinct integers. We see that $\sum\left(\sqrt{n+1}-\sqrt{n}\right)$ fails the Cauchy Criterion and therefore diverges. □

Notes. I didn't intend the $\sum \frac{1}{n^2}$ to be quite so tricky, but it's good to see what sort of cleverness is sometimes involved for such proofs. (Note the Integral Test gives us a much easier proof that $\sum \frac{1}{n^2}$ converges, though we hadn't covered it yet.) Also, really make sure you understand these complicated negations – i.e. what do we need to show to see a sequence isn't Cauchy or doesn't converge. □

UNGRADED HOMEWORK:

Pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises. Note we already did several of the Section 12 exercises together.

Section	Exercises
12	1*, 2, 3, 4*, 5*, 6*, 7, 8*, 10, 14
14	1, 2, 3, 4, 5, 7, 8, 10, 13, 14

* Try doing Exercise 12.11 a different way than we did in class – take the book's proof for the lim sup half and flip it to get the lim inf result, using standard tricks.