MATH 104, HOMEWORK #5 SOLUTIONS

Due Thursday, February 18

Remember, consult the Homework Guidelines for general instructions. Results from class, our textbook, and graded homework are fair game to use unless otherwise specified. You may also use ungraded homework results from previous problem sets.

GRADED HOMEWORK:

1. Let $t_1 = 1$ and $t_{n+1} = t_n \left(1 - \dfrac{1}{(n+1)^2}\right)$ for all $n \geq 1$.

   (a) Explain how the results in Chapter 10 guarantee that $\lim t_n$ exists, though they tell you nothing (or at least very little) about what the limit is.

   (b) Use induction to show that $t_n = \frac{n+1}{2n}$ for all $n \in \mathbb{N}$.

   (c) Find $\lim t_n$ and prove your answer is correct (either by quoting previous results or from scratch).

   Solution. (a) We can easily see that (t_n) is a sequence which is decreasing (as t_{n+1} is obtained by multiplying t_n by a fraction between 0 and 1) and bounded below by 0. It's clearly bounded above by $t_1 = 1$ since it is decreasing. In Section 10, we showed that bounded monotonic sequences converge, so our sequence must converge.

   (b) Base Case: Clearly $t_1 = \frac{1+1}{2 \cdot 1} = 1$.

   Inductive Step. Assume $t_n = \frac{n+1}{2n}$. We wish to show that $t_{n+1} = \frac{(n+1)+1}{2(n+1)} = \frac{n+2}{2(n+1)}$. Using our assumption and the recursive definition of t_{n+1}, we have

   $$t_{n+1} = t_n \left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+1}{2n}\left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+1}{2n}\left(\frac{n(n+2)}{(n+1)^2}\right) = \frac{n+2}{2(n+1)},$$

   as desired.

   (c) Since we proved we have an explicit formula for t_n, we can use the Section 9 limit shortcuts.

   $$\lim t_n = \lim \frac{n+1}{2n} = \lim \frac{n}{2n} + \lim \frac{1}{2n} = \lim \frac{1}{2} + \frac{1}{2} \lim \frac{1}{n} = \frac{1}{2} + 0 = \frac{1}{2},$$

   using the fact that $\lim \frac{1}{n} = 0$. □

   Notes. This one should have been very straightforward. □

2. Let (a_n) be the sequence defined by $a_n = \frac{1}{n}$ if n is odd and $a_n = n$ if n is even. That is $(a_n) = (1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, 8, \ldots)$. Completely characterize (with proof) which subsequences of (a_n) have limits, and determine the set S of subsequential limits of (a_n).

Solution. Claim: A subsequence (a_{n_k}) has $\lim a_{n_k} = +\infty$ if and only if it has infinitely many even-indexed terms and finitely many odd-indexed terms. Similarly, (a_{n_k}) has $\lim a_{n_k} = 0$ if and only if it has finitely many even-indexed terms and infinitely many odd-indexed terms. In every other case, (a_{n_k}) has infinitely many even-indexed terms AND infinitely many odd indexed terms, and its limit does not exist. Our proof of this will show that the set of subsequential limits is precisely $\{0, +\infty\}$.

Proof: If (a_{n_k}) has infinitely many terms $a_{n_k} = n_k$ and only finitely many terms $a_{n_k} = \frac{1}{n_k}$, then its limit must be $+\infty$, as follows. Let N_O denote the largest odd index appearing in (a_{n_k}), so that $a_{n_k} = n_k$ for all $n_k > N_O$. Let $M > 0$ and take $N = \max\{N_O, M\}$. Then $n_k > N$ implies $a_{n_k} = n_k > N \geq M$. This implies $\lim a_{n_k} = +\infty$.

If (a_{n_k}) has finitely many terms $a_{n_k} = n_k$ and infinitely many terms $a_{n_k} = \frac{1}{n_k}$, then its limit must be 0, as follows. Let $\epsilon > 0$ and set $N = \max\{\frac{1}{\epsilon}, N_E\}$, where N_E denotes the largest even index appearing in (a_{n_k}). Then $n_k > N$ implies $a_{n_k} = \frac{1}{n_k}$ and $n_k > \frac{1}{\epsilon}$. Rearranging the latter inequality, we have $\frac{1}{n_k} < \epsilon$, i.e. $|a_{n_k} - 0| < \epsilon$, so $\lim a_{n_k} = 0$.

Note any subsequence must contain infinitely many even-indexed terms or infinitely many odd-indexed terms, otherwise there are only finitely many terms. If a subsequence has infinitely many of both, we see that it diverges. Taking the even-indexed terms only, we get a subsequence (of our subsequence, yes) whose limit is $+\infty$ and taking only the odd-indexed terms, we get another subsequence of (a_{n_k}) converging to 0. Since we have two subsequences of (a_{n_k}) with different limits, we see that $\lim a_{n_k}$ cannot exist, since the limit exists if and only if there is exactly one subsequential limit.

□

Notes. This one probably took a fair amount of experimenting to find the answer, but it was relatively simple in the end. □

3. Create a sequence (b_n) of positive real numbers whose set S of subsequential limits contains infinitely many numbers from the closed interval $[0, 1]$. You may clearly describe your sequence in words or with a recursive definition if you find it difficult to write an explicit formula. Prove that your answer meets the criterion.

Solution. A simple sequence here is $(1, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}, \ldots)$. Since every rational number of the form $\frac{a}{b}$ with $a \leq b$ appears infinitely many times, we can take a constant subsequence with the limit $\frac{a}{b}$. (For example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$, so there is a constant subsequence with every term equal to $\frac{1}{2}$.) Thus S has infinitely many elements. □

Notes. Another one that is easy-ish once you see it. □

* Extra Credit (3 points) – Find the serious mathematical error in the book's proof of Theorem 11.2 (i) and explain how to fix it. (There is a very quick fix.) Include a figure that shows what is going on.

Solution. There is a spot where they used ϵ when it should have been $t - \epsilon$. I'll fill in more details on Piazza if someone asks. $\square$

UNGRADED HOMEWORK:

Pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises.

Section	Exercises
10	3, 5, 10
11	1, 2, 3, 4, 8*, 9, 10, 11

* No really, make sure you know how to prove the Squeeze Theorem for sequences.