MATH 104, HOMEWORK #4

Due Thursday, February 11

Remember, consult the Homework Guidelines for general instructions. Feel free to use any theorems up through Section 10 if they apply, unless the problem specifically says otherwise. However, remember that you need to state any results you use (rather than calling it Theorem 9.2 or whatever).

GRADED HOMEWORK:

1. More work with limits.
   (a) Find $\lim_{n\to\infty} \frac{\log_a n}{n}$ for $a > 1$ and give a formal proof of your answer.
   (b) Find $\lim_{n\to\infty} \frac{\log_a n}{n}$ for $0 < a < 1$ and give a formal proof of your answer.
2. Find the following limit, and give a formal proof of your answer. *(Hint: this would be rather tedious/difficult to do from scratch, so you should apply prior results instead.)*

$$\lim_{n\to\infty} \frac{(-1)^{n+1}17^n}{3^{2n+1}n!}$$

3. Theorem 10.11 (in particular the half which says Cauchy sequences are always convergent) relies heavily on the Completeness Axiom. It really should say that a Cauchy sequence of real numbers always converges in $\mathbb{R}$. There are some other spaces where this result does not hold (essentially because they are not complete).
   (a) Suppose you are working in $\mathbb{Q}$. Find a sequence (a_n) in $\mathbb{Q}$ which is a Cauchy sequence, but which does not converge (to an element of $\mathbb{Q}$). Rigorously justify both claims (Cauchy and non-convergent).
   (b) Suppose you are working in the half-open interval $(0, 1]$. Find a sequence (b_n) in $(0, 1]$ which is a Cauchy sequence, but which does not converge (to an element of $(0, 1]$). Rigorously justify both parts.

UNGRADED HOMEWORK:

Pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises. The Section 9 exercises this week mostly formalize the vague notions we discussed about polynomial, exponential, and factorial growth, plus a couple other familiar sequences.

Section	Exercises
9	13*, 14*, 15*, 16, 17, 18*
10	1, 6, 7, 8, 9, 11

* We proved earlier that if (a_n) converges to a, then $(\sqrt{a_n})$ converges to $\sqrt{a}$ (given appropriate assumptions so that this even makes sense). How would you modify the proof for other roots, e.g. $(\sqrt[5]{a_n})$, or really, any rational exponent?
* Given that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$, prove that $\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k$ for any integer k. (Actually, this works when k is any real number, but the proof is somewhat less tedious for just integers.)