MATH 104, HOMEWORK #4 SOLUTIONS

Due Thursday, February 11

Remember, consult the Homework Guidelines for general instructions. Feel free to use any theorems up through Section 10 if they apply, unless the problem specifically says otherwise. However, remember that you need to state any results you use (rather than calling it Theorem 9.2 or whatever).

GRADED HOMEWORK:

1. More work with limits.

   (a) Find $\lim_{n\to\infty} \frac{\log_a n}{n}$ for $a > 1$ and give a formal proof of your answer.

   (b) Find $\lim_{n\to\infty} \frac{\log_a n}{n}$ for $0 < a < 1$ and give a formal proof of your answer.

   Solution. (a) We will first prove a lemma: if (s_n) is a sequence of positive real numbers with $s_n \to 1$, then $(\log_a(s_n))$ is a sequence converging to 0 for any base $a > 1$. Let $\epsilon > 0$. Set $\epsilon_1 = \min\{a^\epsilon - 1, 1 - a^{-\epsilon}\}$. (Since $a > 1$, both choices for ϵ_1 are positive.) Choose N so that $n > N$ implies $|s_n - 1| < \epsilon_1$, which is possible since $s_n \to 1$. Then we have $-\epsilon_1 < s_n - 1 < \epsilon_1$, which gives us $a^{-\epsilon} - 1 < s_n - 1 < a^\epsilon - 1$ or equivalently, $a^{-\epsilon} < a^{\log_a(s_n)} < a^\epsilon$. Since $a > 1$ and we have exponential growth (rather than decay), we see that $-\epsilon < \log_a(s_n) < \epsilon$, and thus $|\log_a(s_n) - 0| < \epsilon$. Thus $\lim \log_a(s_n) = 0$.

   Then, to finish the proof, recall that we showed in the book that $\lim n^{\frac{1}{n}} = 1$. We can rewrite
   $$\frac{\log_a n}{n} = \frac{1}{n}\log_a(n) = \log_a(n^{\frac{1}{n}}).$$
   Thus, if we take $s_n = n^{\frac{1}{n}}$ in the notation of our lemma, we see that $\lim \frac{\log_a n}{n} = 0$.

   (b) For the second portion, notice that if $0 < a < 1$, then $\frac{1}{a} > 1$. Also, $\log_a(n) = -\log_{\frac{1}{a}}(n)$ by basic logarithm properties. Using our limit shortcut for pulling out constant factors, we have
   $$\lim_{n\to\infty} \frac{\log_a n}{n} = \lim_{n\to\infty} -\frac{\log_{\frac{1}{a}} n}{n} = -\lim_{n\to\infty} \frac{\log_{\frac{1}{a}} n}{n} = 0,$$
   with the final conclusion due to part (a). □

   Notes. It turns out the proof for this lemma is noticeably harder if you have a different limit for s_n, but we'll see it with some more general results later.

   On this one you likely ran into a lot of issues on what sort of material is fair game to use for this class, and it's a very gray area. It's something I am constantly thinking about; we'll just keep discussing. □

2. Find the following limit, and give a formal proof of your answer. *(Hint: this would be rather tedious/difficult to do from scratch, so you should apply prior results instead.)*
   $$\lim_{n\to\infty} \frac{(-1)^{n+1}17^n}{3^{2n+1}n!}$$

Solution. We claim that this limit is 0. To show it, we will use the result from Problem 2 from HW 3: if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then $\lim_{n\to\infty} a_n = 0$.

If we take (a_n) to be the sequence defined above, we have

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}17^{n+1}}{3^{2(n+1)+1}(n+1)!} \cdot \frac{3^{2n+1}n!}{(-1)^{n+1}17^n} = \frac{-17}{9(n+1)}.$$

Using our limit shortcut theorems, we have $\lim \frac{-17}{9(n+1)} = \frac{-17}{9} \lim \frac{1}{n+1}$. Since $0 \le \frac{1}{n+1} \le \frac{1}{n}$ for all positive integers n, and $0 = \lim 0 = \lim \frac{1}{n}$, the Squeeze Theorem implies that $\lim \frac{1}{n+1} = 0$, so $\lim \frac{-17}{9(n+1)} = \frac{-17}{9} \cdot 0 = 0 < 1$. Finally, HW 2, Problem 2 quoted above implies that $\lim a_n = 0$. □

Notes. Instead of using the Squeeze Theorem, you may have showed that $\lim \frac{-17}{9(n+1)} = 0$ from scratch. The natural choice for N here would be $N = \max\{0, \frac{17}{9\epsilon} - 1\}$. (To keep things clear, it's good to make sure N is not negative, which is why we take the max with 0.) □

3. Theorem 10.11 (in particular the half which says Cauchy sequences are always convergent) relies heavily on the Completeness Axiom. It really should say that a Cauchy sequence of real numbers always converges in $\mathbb{R}$. There are some other spaces where this result does not hold (essentially because they are not complete).

   (a) Suppose you are working in $\mathbb{Q}$. Find a sequence (a_n) in $\mathbb{Q}$ which is a Cauchy sequence, but which does not converge (to an element of $\mathbb{Q}$). Rigorously justify both claims (Cauchy and non-convergent).

   (b) Suppose you are working in the half-open interval $(0, 1]$. Find a sequence (b_n) in $(0, 1]$ which is a Cauchy sequence, but which does not converge (to an element of $(0, 1]$). Rigorously justify both parts.

Solution. (1) Answers may vary quite a bit here. One relatively simple sequence to work with is increasingly accurate decimal approximations of $\sqrt{2}$, which we know is irrational from previous sections. One way to make this precise is to set $a_n = \frac{\lfloor 10^n\sqrt{2}\rfloor}{10^n}$. Here $a_0 = 1$, i.e. $\sqrt{2}$ rounded down to the nearest integer, and then $a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$, picking up the next digit each time, and so on. Then it is easy to check that if $n, m > N$, then $|a_n - a_m| < \frac{1}{10^N}$. (For example, if you keep at least two digits after the decimal, then no two later terms have a distance more than $\frac{1}{100}$ from each other.) Since each $\epsilon > 0$ is larger than some 10^N, taking such an N each time will show us we have a Cauchy sequence.

(b) Here the only reasonable thing to do is take a sequence limiting to 0. One such sequence is $b_n = \frac{1}{n}$. We know that $\lim \frac{1}{n} = 0$ from previous work, so it does not converge in the interval $(0, 1]$, since 0 is not in this interval. To show it is a Cauchy sequence, note that b_n is a strictly decreasing sequence bounded below by 0. Thus for $m, n > N$, we have $|b_n - b_m| < \frac{1}{N}$. Since each $\epsilon > 0$ is larger than some $\frac{1}{N}$, this means our sequence is Cauchy. □

Notes. One critical part here is making sure you can justify that a number is not in $\mathbb{Q}$. Several people thought of using the sequence limiting to e below, but we don't actually have a good explanation for why e is irrational yet. Furthermore, it would be extremely tough to directly show that that sequence is Cauchy.

What I intended here, though I should have specified more clearly, is for you to practice showing a sequence is Cauchy from the definition. Make sure you can do that. $\square$

UNGRADED HOMEWORK:

Pay special attention to starred problems; they are usually classics we will use many times, often important theorems hidden in the exercises. The Section 9 exercises this week mostly formalize the vague notions we discussed about polynomial, exponential, and factorial growth, plus a couple other familiar sequences.

Section	Exercises
9	13*, 14*, 15*, 16, 17, 18*
10	1, 6, 7, 8, 9, 11

* We proved earlier that if (a_n) converges to a, then $(\sqrt{a_n})$ converges to $\sqrt{a}$ (given appropriate assumptions so that this even makes sense). How would you modify the proof for other roots, e.g. $(\sqrt[5]{a_n})$, or really, any rational exponent?

* Given that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, prove that $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$ for any integer k. (Actually, this works when k is any real number, but the proof is somewhat less tedious for just integers.)